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Error Bounds of Boole's Formula for Twice Differentiable Convex Functions with Numerical Analysis

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Abstract

This study investigates the error bounds associated with Boole's formula, a well-known open Newton–Cotes technique. By focusing on twice differentiable functions, a novel identity for Boole's formula is established. Establishing a new identity is a challenging task in this research because suitable kernels need to be found by replacing unknown values. The number of equations is greater than the number of unknowns, so the system has infinitely many solutions, and only one is suitable for Boole's type identity. Using the newly established equality, several Boole's type inequalities for twice differentiable convex functions are demonstrated. It offers significant advancements in the domain of computational mathematics and its applications. Additionally, numerical examples and graphical illustrations are included to validate the significance and applicability of these newly established inequalities.

Keywords: Inequalities of Boole's rule, error bound, twice differentiable convex function.

MSC2020: 26D15, 26D10, 26D07

1 Introduction and Preliminaries

Numerical integration, a key concept in computational mathematics, is utilized to solve an extensive number of mathematical problems and approximate definite integrals. In 1915, the term "numerical integration" first appeared in the publication under the title "A Course in Interpolation and Numerical Integration for the Mathematical Laboratory" by David Gibb [6]. In recent decades, numerical integration has become an essential tool in scientific computing, engineering, and data analysis. To address modern approaches like adaptive quadrature, integration methods with error control, and strategies for high-dimensional integrals have been introduced to tackle more complex computational tasks. Quadrature is an old mathematical term in numerical integration that means calculating area. By constructing different interpolating polynomials, a large class of quadrature rules can be derived. One of the simplest methods of this type is to let the interpolating function be constant (zero degrees polynomial) that passes through the point $\left(\frac{\kappa+\aleph}{2}, f\left(\frac{\kappa+\aleph}{2}\right)\right)$. This is called the midpoint rule or rectangle rule

$$\int_{\kappa}^{\aleph} f(x)\,dx = (\aleph-\kappa)\, f\left(\frac{\kappa+\aleph}{2}\right) + E(f). \tag{1}$$

The interpolating function may be a straight line (a linear polynomial) passing through the points $(\kappa, f(\kappa))$ and $(\aleph, f(\aleph))$. This is known as the trapezoidal rule

$$\int_{\kappa}^{\aleph} f(x)\,dx = (\aleph-\kappa)\left(\frac{f(\kappa)+f(\aleph)}{2}\right) + E(f). \tag{2}$$

We get Simpson's rule if the interpolating polynomial is of second degree. It is named Simpson's rule due to mathematician Thomas Simpson (1710-1761). The most basic of Simpson's rule is Simpson's 1/3 rule

$$\int_{\kappa}^{\aleph} f(x)\,dx = \frac{(\aleph-\kappa)}{6}\left[f(\kappa) + 4f\left(\frac{\kappa+\aleph}{2}\right) + f(\aleph)\right] + E(f). \tag{3}$$

Another Simpson rule called the Simpson's 3/8 rule or Second Simpson's rule. It requires one more function evaluation with lower error bounds but does not improve the order of the error

$$\int_{\kappa}^{\aleph} f(x)\,dx = \frac{(\aleph-\kappa)}{8}\left[f(\kappa) + 3f\left(\frac{2\kappa+\aleph}{3}\right) + 3f\left(\frac{\kappa+2\aleph}{3}\right) + f(\aleph)\right] + E(f). \tag{4}$$

In numerical integration with lower error bounds, the Third Simpson's rule or Simpson's 2/45 rule, also called Boole's rule, is named after George Boole

$$\int_{\kappa}^{\aleph} f(x)\,dx = \frac{2h}{45}\left[7f(\kappa) + 32f\left(\frac{3\kappa+\aleph}{4}\right) + 12f\left(\frac{\kappa+\aleph}{2}\right) + 32f\left(\frac{\kappa+3\aleph}{4}\right) + 7f(\aleph)\right] + E(f), \tag{5}$$

where $h = (\aleph-\kappa)/4$ and $E(f)$ is an error term. For more comprehensive discussions on numerical integration one can visit [18, 5] for numerical integration and its applications.

To describe the importance of $E(f)$, we need to discuss error analysis. From an applied perspective, the utility of an approximation lies in understanding its accuracy. For any computed integral approximation, the error represents the disparity between the approximation and the exact value of the integral. Suppose a function is twice differentiable everywhere on the interval $[\kappa, \aleph]$ and bounded. The error bound for midpoint approximation is defined as:

$$E(f) = \frac{(\aleph-\kappa)^3}{24}M,\ |f''(x)| \le M. \tag{6}$$

The subsequent expression represents the error bound for trapezoidal rule approximation:

$$E(f) = \frac{(\aleph-\kappa)^3}{12}M,\ |f''(x)| \le M. \tag{7}$$

The error bound for Simpson's rule approximation is described as:

$$E(f) = -\frac{(\aleph-\kappa)^5}{2880}M,\ \left|f^{(iv)}(x)\right| \le M. \tag{8}$$

The error bound for Boole's rule approximation is described as:

$$E(f) = -\frac{8(\aleph - \kappa)^8}{945} M, \ \left|f^{(vi)}(x)\right| \leq M. \tag{9}$$

In (6) - (9), we have an error term $E(f)$ for the above-described methods, but the function should be a twice differentiable and bounded. In 1998, to overcome this issue, Dragomir and Agarwal provided an error term $E(f)$. They showed that the remainder term can be estimated in terms of the first derivative only for the trapezoidal rule [3]. After studying Dragomir's article, Kirmaci provided an error term of the midpoint rule in the same pattern in 2004 [14]. In 2010, Sarikaya et al. proved Simpson-type inequalities for differentiable convex functions and found their error bounds [26]. We need the convexity of first-time differentiable functions to estimate the error term of the above-described methods, which is a more significant class of functions than that of bounded two-times differentiable functions. That is why this is a considerable achievement in inequality theory.

In mathematical analysis, inequality theory serves as crucial because it establishes bounds and provides important insights into how functions behave in various domains. In the context of optimization, inequalities are particularly essential as they establish necessary conditions for optimal solutions. Inequality theory also has a significant impact on numerical approaches, helping to improve the accuracy of integral approximations and develop error estimates. It improves the efficacy and reliability of models and algorithms in a variety of fields, such as computer science, engineering, and economics. The study of classical inequalities applied to integral operators associated with various fractional integrals has increased significantly in recent years. A few examples of these investigated inequalities are Jensen, Gruss, Hölder, Hermite–Hadamard, Simpson, Euler-Maclaurin, and Milne-type [20, 27, 22].

Dragomir and Agarwal's findings regarding the estimation of errors associated with the trapezoidal formula are reported in [3]. Kirmaci defined error bounds for the trapezoidal and midpoint formulas by utilizing convex functions [13]. With the objective to derive new generalized inequalities of the Hermite–Hadamard type, Sarikaya et al. suggested a novel integral identity for fractional integrals [25]. Tunç and Demir [29] have introduced a new identity for the proportional Caputo-hybrid operator and obtained some new integral inequalities related to the right-hand side of the Hermite–Hadamard type inequalities. In [17] authors investigated symmetric four-point Newton–Cotes-type inequalities and obtained error estimates for various function classes. Meftah et al. [21] have established Hermite–Hadamard type inequalities for functions whose second derivatives are s-convex. Also, they obtained several inequalities for twice differantiable mappings. Hwang et al. [11] identified several new improvements and similar generalizations of Hermite–Hadamard inequalities for fractional integrals and provide certain applications in the Beta function. For further study of inequalities involving fractional integrals, consult [24, 12, 31] and references therein.

In [16], Krukowski studied the logical derivation of Simpson's rule, which is in Boole's rule. Taking advantage of computer software, new error bounds for Boole's rule have been derived. Dragomir et al. [4] uncovered novel Simpson-type inequalities. Further, they examine their applications to quadrature formulas. Ali et al. [1] have formulated generalized fractional integral equality employing twice differentiable functions. They identified several novel Simpson-type inequalities for twice differentiable convex functions. For twice differentiable geometric-arithmetically s-convex functions, Liao et al. [19] have offered Hermite–Hadamard inequalities by taking advantage of the beta function and incomplete beta function. In the context of generalized, You et al. [32] have provided a novel equality for twice differentiable functions. Also, they examined some Simpson-type inequalities whose second derivatives in absolute values are convex. Shehzadi et al. [28] derived a novel identity by generalizing Boole's rule for the first time. They investigated several results related to differentiable convex functions and studied different function classes such as bounded function, Lipschitzian function, and functions with bounded variation.

Studying twice differentiable convex functions allows for a detailed analysis of how errors behave as the function changes or as the interval size changes. It can help in adaptive quadrature methods, where the partition of the interval is adjusted based on the function's behaviour. Some important articles that encouraged us for twice differentiable convex functions such as Simpson-type [10], corrected Euler–Maclaurin-type [8], Milne-type [7], Hermite–Hadamard-type for h-Convex functions [30], Bullen-type [9], and so on [23, 2, 15].

Inspired by ongoing investigations, we aim to establish novel Boole's type inequalities for twice differentiable convex functions. Initially, we formulate a new identity for Boole's type concerning twice differentiable functions. Establishing a new identity is a challenging task in this research to find suitable kernels by replacing unknown constant. The number of equations is greater than the number of unknowns, so the system has many solutions, and only one is suitable for Boole's type identity. By taking advantage of this equality, we derived several new inequalities in the framework of twice differentiable convex functions. Furthermore, we include numerical examples and graphical representations as well as a comparison with previously proven results that demonstrate the accuracy of the newly obtained inequalities.

The study contains five sections, commencing with the introduction and preliminaries, which include basic definitions and a summary of pertinent research in the field. We will demonstrate various results related to Boole's type inequality for twice differentiable convex functions in Section 2. Section 3, delves into applications of the proposed inequalities to quadrature formulas and special means, showcasing their practical significance. In Section 4, we will provide examples to clarify these inequalities, emphasizing their practical applications and significance. In the final section, we summarize our results and offer possible avenues for future study.

2 Principal Finding

In this section, we utilize twice differentiable convex mappings to derive a sequence of Boole's-type inequalities. In order to accomplish this, we initially provide proof of that specific identity. Consequently, we derive three distinct inequalities by employing the modulus of this identity. This strategy not only presents new results but also enhances the advancement of the field.

Lemma 2.1. *Assume that $f : [\kappa, \aleph] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function on $(\kappa, \aleph)$. If $f'' \in L[\kappa, \aleph]$, then, we have the subsequent equality:*

$$\frac{1}{\aleph-\kappa}\int_{\kappa}^{\aleph} f(u)\,du - \frac{1}{90}\left[7f(\kappa) + 32f\left(\frac{3\kappa+\aleph}{4}\right) + 12f\left(\frac{\kappa+\aleph}{2}\right) + 32f\left(\frac{\kappa+3\aleph}{4}\right) + 7f(\aleph)\right]$$
$$= \frac{(\aleph-\kappa)^2}{2}\sum_{j=0}^{3} S_j, \tag{10}$$

where

$$S_0 = \int_0^{\frac{1}{4}} \left(\varphi^2 - \frac{7}{45}\varphi\right) f''(\varphi\aleph + (1-\varphi)\kappa)\,d\varphi,$$
$$S_1 = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\varphi^2 - \frac{13}{15}\varphi + \frac{8}{45}\right) f''(\varphi\aleph + (1-\varphi)\kappa)\,d\varphi,$$
$$S_2 = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\varphi^2 - \frac{17}{15}\varphi + \frac{14}{45}\right) f''(\varphi\aleph + (1-\varphi)\kappa)\,d\varphi,$$

$$S_3 = \int_{\frac{3}{4}}^{1} \left(\varphi^2 - \frac{83}{45}\varphi + \frac{38}{45}\right) f''\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi.$$

Proof. Taking right hand side of (10), we have

$$\frac{(\aleph-\kappa)^2}{2}\sum_{j=0}^{3} S_j = \frac{(\aleph-\kappa)^2}{2}\left[S_0 + S_1 + S_2 + S_3\right].$$

Using the integration by parts, it yields

$$\begin{aligned}
S_0 &= \int_0^{\frac{1}{4}} \left(\varphi^2 - \frac{7}{45}\varphi\right) f''\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi \\
&= \frac{1}{\aleph-\kappa}\left[\left(\varphi^2 - \frac{7}{45}\varphi\right) f'\left(\varphi\aleph + (1-\varphi)\kappa\right)\Big|_0^{\frac{1}{4}} - \int_0^{\frac{1}{4}} \left(2\varphi - \frac{7}{45}\right) f'\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi\right] \\
&= \frac{1}{\aleph-\kappa}\left(\frac{17}{720}\right) f'\left(\frac{3\kappa+\aleph}{4}\right) - \frac{1}{(\aleph-\kappa)^2}\left[\left(2\varphi - \frac{7}{45}\right) f\left(\varphi\aleph + (1-\varphi)\kappa\right)\Big|_0^{\frac{1}{4}}\right. \\
&\quad \left. -2\int_0^{\frac{1}{4}} f\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi\right] \\
&= \frac{1}{\aleph-\kappa}\left(\frac{17}{720}\right) f'\left(\frac{3\kappa+\aleph}{4}\right) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{31}{90}\right) f\left(\frac{3\kappa+\aleph}{4}\right) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{7}{45}\right) f(\kappa) \\
&\quad + \frac{2}{(\aleph-\kappa)^2}\int_0^{\frac{1}{4}} f\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi.
\end{aligned} \tag{11}$$

In a similar manner, we acquire

$$\begin{aligned}
S_1 &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\varphi^2 - \frac{13}{15}\varphi + \frac{8}{45}\right) f''\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi \\
&= \frac{1}{\aleph-\kappa}\left(-\frac{1}{180}\right) f'\left(\frac{\kappa+\aleph}{2}\right) - \frac{1}{\aleph-\kappa}\left(\frac{17}{720}\right) f'\left(\frac{3\kappa+\aleph}{4}\right) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{2}{15}\right) f\left(\frac{\kappa+\aleph}{2}\right) \\
&\quad - \frac{1}{(\aleph-\kappa)^2}\left(\frac{11}{30}\right) f\left(\frac{3\kappa+\aleph}{4}\right) + \frac{2}{(\aleph-\kappa)^2}\int_{\frac{1}{4}}^{\frac{1}{2}} f\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi,
\end{aligned} \tag{12}$$

$$\begin{aligned}
S_2 &= \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\varphi^2 - \frac{17}{15}\varphi + \frac{14}{45}\right) f''\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi \\
&= \frac{1}{\aleph-\kappa}\left(\frac{17}{720}\right) f'\left(\frac{\kappa+3\aleph}{4}\right) + \frac{1}{\aleph-\kappa}\left(\frac{1}{180}\right) f'\left(\frac{\kappa+\aleph}{2}\right) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{11}{30}\right) f\left(\frac{\kappa+3\aleph}{4}\right) \\
&\quad - \frac{1}{(\aleph-\kappa)^2}\left(\frac{2}{15}\right) f\left(\frac{\kappa+\aleph}{2}\right) + \frac{2}{(\aleph-\kappa)^2}\int_{\frac{1}{4}}^{\frac{1}{2}} f\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi,
\end{aligned} \tag{13}$$

and

$$S_3 = \int_{\frac{3}{4}}^{1} \left(\varphi^2 - \frac{83}{45}\varphi + \frac{38}{45}\right) f''\left(\varphi\aleph + (1-\varphi)\kappa\right) d\varphi$$

$$= \frac{1}{\aleph-\kappa}\left(-\frac{17}{720}\right) f'\left(\frac{\kappa+3\aleph}{4}\right) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{7}{45}\right) f(\aleph) - \frac{1}{(\aleph-\kappa)^2}\left(\frac{31}{90}\right) f\left(\frac{\kappa+3\aleph}{4}\right)$$
$$+ \frac{2}{(\aleph-\kappa)^2}\int_{\frac{3}{4}}^{1} f(\varphi\aleph+(1-\varphi)\kappa)\,d\varphi. \tag{14}$$

Adding (11)-(14), multiplying the resulting equality by $\frac{(\aleph-\kappa)^2}{2}$, then by changing of variable we get desired result. Hence, proof of Lemma 2.1, is completed. □

Theorem 2.2. *Assume that $f:[\kappa,\aleph]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function on $(\kappa,\aleph)$, such that $f''\in L[\kappa,\aleph]$. If $|f''|$ is convex on $[\kappa,\aleph]$, then the subsequent inequality holds:*

$$\left|\frac{1}{\aleph-\kappa}\int_{\kappa}^{\aleph} f(u)\,du - \frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|$$
$$\leq \frac{509(\aleph-\kappa)^2}{273375}\left[|f''(\kappa)|+|f''(\aleph)|\right]. \tag{15}$$

Proof. For the absolute value of Lemma 2.1, we have

$$\left|\frac{1}{\aleph-\kappa}\int_{\kappa}^{\aleph} f(u)\,du - \frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|$$
$$\leq \frac{(\aleph-\kappa)^2}{2}\left\{\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right| |f''(\varphi\aleph+(1-\varphi)\kappa)|\,d\varphi + \int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right| |f''(\varphi\aleph+(1-\varphi)\kappa)|\,d\varphi\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right| |f''(\varphi\aleph+(1-\varphi)\kappa)|\,d\varphi + \int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right| |f''(\varphi\aleph+(1-\varphi)\kappa)|\,d\varphi\right\}. \tag{16}$$

By using convexity of $|f''|$, we attain

$$\left|\frac{1}{\aleph-\kappa}\int_{\kappa}^{\aleph} f(u)\,du - \frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|$$
$$\leq \frac{(\aleph-\kappa)^2}{2}\left\{\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|[\varphi|f''(\aleph)|+(1-\varphi)|f''(\kappa)|]\,d\varphi\right.$$
$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|[\varphi|f''(\aleph)|+(1-\varphi)|f''(\kappa)|]\,d\varphi$$
$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|[\varphi|f''(\aleph)|+(1-\varphi)|f''(\kappa)|]\,d\varphi$$
$$\left.+\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|[\varphi|f''(\aleph)|+(1-\varphi)|f''(\kappa)|]\,d\varphi\right\}$$
$$= \frac{(\aleph-\kappa)^2}{2}\left\{[\zeta_1(\varphi)+\zeta_2(\varphi)+\zeta_3(\varphi)+\zeta_4(\varphi)]\,|f''(\kappa)|\right.$$
$$\left.+[\mu_1(\varphi)+\mu_2(\varphi)+\mu_3(\varphi)+\mu_4(\varphi)]\,|f''(\aleph)|\right\},$$

where

$$\zeta_1(\varphi)=\int_0^{\frac14}\left|\varphi^2-\frac{7}{45}\varphi\right|\varphi d\varphi=\frac{3325187}{12597120000},$$

$$\zeta_2(\varphi)=\int_{\frac14}^{\frac12}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|\varphi d\varphi=\frac{961}{1244160},$$

$$\zeta_3(\varphi)=\int_{\frac12}^{\frac34}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|\varphi d\varphi=\frac{1679}{1244160},$$

$$\zeta_4(\varphi)=\int_{\frac34}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|\varphi d\varphi=\frac{16854253}{12597120000},$$

$$\mu_1(\varphi)=\int_0^{\frac14}\left|\varphi^2-\frac{7}{45}\varphi\right|(1-\varphi) d\varphi=\frac{16854253}{12597120000},$$

$$\mu_2(\varphi)=\int_0^{\frac14}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|(1-\varphi) d\varphi=\frac{1679}{1244160},$$

$$\mu_3(\varphi)=\int_{\frac12}^{\frac34}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|(1-\varphi) d\varphi=\frac{961}{1244160},$$

$$\mu_4(\varphi)=\int_{\frac34}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|(1-\varphi) d\varphi=\frac{3325187}{12597120000}.$$

Hence, the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Assume that $f:[\kappa,\aleph]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function on $(\kappa,\aleph)$, such that $f''\in L[\kappa,\aleph]$. If $|f''|^q$ is convex on $[\kappa,\aleph]$ with $q>1$, then the subsequent inequality holds:*

$$\begin{aligned}&\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|\\
&\le\frac{(\aleph-\kappa)^2}{2}\left[\left(\frac{28027}{17496000}\right)^{1-\frac1q}\left(\frac{3325187|f''(\aleph)|^q+16854253|f''(\kappa)|^q}{12597120000}\right)^{\frac1q}\right.\\
&\quad+\left(\frac{11}{5184}\right)^{1-\frac1q}\left(\frac{961|f''(\aleph)|^q+1679|f''(\kappa)|^q}{1244160}\right)^{\frac1q}+\left(\frac{11}{5184}\right)^{1-\frac1q}\left(\frac{1679|f''(\aleph)|^q+961|f''(\kappa)|^q}{1244160}\right)^{\frac1q}\\
&\quad\left.+\left(\frac{28027}{17496000}\right)^{1-\frac1q}\left(\frac{16854253|f''(\aleph)|^q+3325187|f''(\kappa)|^q}{12597120000}\right)^{\frac1q}\right].\end{aligned}\tag{17}$$

Proof. When we first apply (16) to the power-mean inequality, it becomes

$$\begin{aligned}&\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|\\
&\le\frac{(\aleph-\kappa)^2}{2}\left\{\left(\int_0^{\frac14}\left|\varphi^2-\frac{7}{45}\varphi\right|d\varphi\right)^{1-\frac1q}\left(\int_0^{\frac14}\left|\varphi^2-\frac{7}{45}\varphi\right||f''(\varphi\aleph+(1-\varphi)\kappa)|^q d\varphi\right)^{\frac1q}\right.\\
&\quad+\left(\int_{\frac14}^{\frac12}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|d\varphi\right)^{1-\frac1q}\left(\int_{\frac14}^{\frac12}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right||f''(\varphi\aleph+(1-\varphi)\kappa)|^q d\varphi\right)^{\frac1q}\end{aligned}$$

$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|\left|f''\left(\varphi\aleph+(1-\varphi)\kappa\right)\right|^q d\varphi\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|\left|f''\left(\varphi\aleph+(1-\varphi)\kappa\right)\right|^q d\varphi\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|\left|f''\left(\varphi\aleph+(1-\varphi)\kappa\right)\right|^q d\varphi\right)^{\frac{1}{q}}\right\}.$$

As $|f''|^q$ is convex on the interval $[\kappa,\aleph]$, we attain

$$\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|$$
$$\leq\frac{(\aleph-\kappa)^2}{2}\left\{\left(\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right]d\varphi\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right]d\varphi\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right]d\varphi\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right]d\varphi\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(\aleph-\kappa)^2}{2}\left[\left(\frac{28027}{17496000}\right)^{1-\frac{1}{q}}\left(\frac{3325187|f''(\aleph)|^q+16854253|f''(\kappa)|^q}{12597120000}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{11}{5184}\right)^{1-\frac{1}{q}}\left(\frac{961|f''(\aleph)|^q+1679|f''(\kappa)|^q}{1244160}\right)^{\frac{1}{q}}+\left(\frac{11}{5184}\right)^{1-\frac{1}{q}}\left(\frac{1679|f''(\aleph)|^q+961|f''(\kappa)|^q}{1244160}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{28027}{17496000}\right)^{1-\frac{1}{q}}\left(\frac{16854253|f''(\aleph)|^q+3325187|f''(\kappa)|^q}{12597120000}\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Assume that $f:[\kappa,\aleph]\subset\mathbb{R}\to\mathbb{R}$ is a twice differentiable function on $(\kappa,\aleph)$, such that $f''\in L[\kappa,\aleph]$. If $|f''|^q$ is convex on $[\kappa,\aleph]$, then the subsequent inequality holds:*

$$\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|$$
$$\leq\frac{(\aleph-\kappa)^2}{2}\left[\Omega_1^p\left(\frac{|f''(\aleph)|^q+7|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}+\Omega_2^p\left(\frac{3|f''(\aleph)|^q+5|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}\right.$$
$$\left.+\Omega_3^p\left(\frac{5|f''(\aleph)|^q+3|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}+\Omega_4^p\left(\frac{7|f''(\aleph)|^q+|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}\right],\tag{18}$$

where $q^{-1}+p^{-1}=1$ *and*

$$\Omega_1^p=\left(\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|^p d\varphi\right)^{\frac{1}{p}},$$

$$\Omega_2^p=\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|^p d\varphi\right)^{\frac{1}{p}},$$

$$\Omega_3^p=\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|^p d\varphi\right)^{\frac{1}{p}},$$

$$\Omega_4^p=\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|^p d\varphi\right)^{\frac{1}{p}}.$$

Proof. By utilizing Hölder's inequality in (16), it yields

$$\begin{aligned}&\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|\\
&\leq\frac{(\aleph-\kappa)^2}{2}\left\{\left(\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}\left|f''(\varphi\aleph+(1-\varphi)\kappa)\right|^q d\varphi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|f''(\varphi\aleph+(1-\varphi)\kappa)\right|^q d\varphi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|f''(\varphi\aleph+(1-\varphi)\kappa)\right|^q d\varphi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left|f''(\varphi\aleph+(1-\varphi)\kappa)\right|^q d\varphi\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|f''|^q$ is convex on the interval $[\kappa,\aleph]$, it becomes

$$\begin{aligned}&\left|\frac{1}{\aleph-\kappa}\int_\kappa^\aleph f(u)\,du-\frac{1}{90}\left[7f(\kappa)+32f\left(\frac{3\kappa+\aleph}{4}\right)+12f\left(\frac{\kappa+\aleph}{2}\right)+32f\left(\frac{\kappa+3\aleph}{4}\right)+7f(\aleph)\right]\right|\\
&\leq\frac{(\aleph-\kappa)^2}{2}\left\{\left(\int_0^{\frac{1}{4}}\left|\varphi^2-\frac{7}{45}\varphi\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right] d\varphi\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\varphi^2-\frac{13}{15}\varphi+\frac{8}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right] d\varphi\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\varphi^2-\frac{17}{15}\varphi+\frac{14}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right] d\varphi\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{3}{4}}^{1}\left|\varphi^2-\frac{83}{45}\varphi+\frac{38}{45}\right|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left[\varphi|f''(\aleph)|^q+(1-\varphi)|f''(\kappa)|^q\right] d\varphi\right)^{\frac{1}{q}}\right\}\end{aligned}$$

$$= \frac{(\aleph-\kappa)^2}{2}\left[\Omega_1^p\left(\frac{|f''(\aleph)|^q+7|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}+\Omega_2^p\left(\frac{3|f''(\aleph)|^q+5|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}\right.$$
$$\left.+\Omega_3^p\left(\frac{5|f''(\aleph)|^q+3|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}+\Omega_4^p\left(\frac{7|f''(\aleph)|^q+|f''(\kappa)|^q}{32}\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 2.4 is completed. □

3 Applications in Numerical Integration

3.1 Application to Boole's Quadrature Formula

Proposition 3.1. *Let $\Upsilon : \kappa = \varkappa_0 < \varkappa_1 < \cdots < \varkappa_n = \aleph$ be a partition of the interval $[\kappa, \aleph]$. Assume that $f : [\kappa, \aleph] \to \mathbb{R}$ is twice differentiable on $(\kappa, \aleph)$, $f'' \in L[\kappa, \aleph]$, and $|f''|$ is convex on $[\kappa, \aleph]$. Then*

$$\int_\kappa^\aleph f(u)\,du = \mathcal{S}(f,\Upsilon) + R(f,\Upsilon),$$

where

$$\mathcal{S}(f,\Upsilon) = \sum_{j=0}^{n-1}\frac{\varkappa_{j+1}-\varkappa_j}{90}\left[7f(\varkappa_j)+32f\left(\frac{3\varkappa_j+\varkappa_{j+1}}{4}\right)\right.$$
$$\left.+12f\left(\frac{\varkappa_j+\varkappa_{j+1}}{2}\right)+32f\left(\frac{\varkappa_j+3\varkappa_{j+1}}{4}\right)+7f(\varkappa_{j+1})\right],$$

and the remainder term satisfies

$$|R(f,\Upsilon)| \le \frac{509}{273375}\sum_{j=0}^{n-1}(\varkappa_{j+1}-\varkappa_j)^3\left[|f''(\varkappa_j)|+|f''(\varkappa_{j+1})|\right].$$

Proof. Applying Theorem 2.2 on each subinterval $[\varkappa_j, \varkappa_{j+1}]$, for $j = 0, 1, \ldots, n-1$, we obtain

$$\left|\frac{1}{\varkappa_{j+1}-\varkappa_j}\int_{\varkappa_j}^{\varkappa_{j+1}} f(u)\,du - \frac{1}{90}\left[7f(\varkappa_j)+32f\left(\frac{3\varkappa_j+\varkappa_{j+1}}{4}\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{\varkappa_j+\varkappa_{j+1}}{2}\right)+32f\left(\frac{\varkappa_j+3\varkappa_{j+1}}{4}\right)+7f(\varkappa_{j+1})\right]\right|$$
$$\le \frac{509(\varkappa_{j+1}-\varkappa_j)^2}{273375}\left[|f''(\varkappa_j)|+|f''(\varkappa_{j+1})|\right].$$

Multiplying both sides by $\varkappa_{j+1}-\varkappa_j$, we get

$$\left|\int_{\varkappa_j}^{\varkappa_{j+1}} f(u)\,du - \frac{\varkappa_{j+1}-\varkappa_j}{90}\left[7f(\varkappa_j)+32f\left(\frac{3\varkappa_j+\varkappa_{j+1}}{4}\right)\right.\right.$$
$$\left.\left.+12f\left(\frac{\varkappa_j+\varkappa_{j+1}}{2}\right)+32f\left(\frac{\varkappa_j+3\varkappa_{j+1}}{4}\right)+7f(\varkappa_{j+1})\right]\right|$$
$$\le \frac{509(\varkappa_{j+1}-\varkappa_j)^3}{273375}\left[|f''(\varkappa_j)|+|f''(\varkappa_{j+1})|\right].$$

Summing over $j = 0, 1, \ldots, n-1$ and using the triangle inequality, we obtain

$$|R(f, \Upsilon)| \leq \frac{509}{273375} \sum_{j=0}^{n-1} (\varkappa_{j+1} - \varkappa_j)^3 \left[|f''(\varkappa_j)| + |f''(\varkappa_{j+1})| \right].$$

This completes the proof. □

3.2 Application to Special Means

We shall consider the following special means. For arbitrary real numbers $\kappa, \varkappa_1, \varkappa_2, \ldots, \varkappa_n, \aleph$, we have

The arithmetic mean: $\mathcal{A} = \mathcal{A}(\varkappa_1, \varkappa_2, \ldots, \varkappa_n) = \frac{\varkappa_1 + \varkappa_2 + \ldots + \varkappa_n}{n}$.

The harmonic mean: $\mathcal{H} = \mathcal{H}(\varkappa_1, \varkappa_2) = \frac{2}{\frac{1}{\varkappa_1} + \frac{1}{\varkappa_2}}, \quad \varkappa_1, \varkappa_2 > 0.$

The logarithm mean: $\mathsf{L} = \mathsf{L}(\varkappa_1, \varkappa_2) = \frac{\varkappa_2 - \varkappa_1}{\ln \varkappa_2 - \ln \varkappa_1}, \qquad \varkappa_1 \neq \varkappa_2, \quad \varkappa_1, \varkappa_2 \in \mathbb{R}^+.$

The p-logarithmic mean: $\mathsf{L}_p(\varkappa_1, \varkappa_2) = \left(\frac{\varkappa_2^{p+1} - \varkappa_1^{p+1}}{(p+1)(\varkappa_2 - \varkappa_1)} \right)^{\frac{1}{p}}, \quad \varkappa_1, \varkappa_2 > 0, \quad \varkappa_1 \neq \varkappa_2$ and $p \in \mathbb{R}\{-1, 0\}$.

Based on the outcomes presented in Section 2, we establish new inequalities for the means mentioned above.

Proposition 3.2. *Let $0 < \kappa < \aleph$ and $n \in \mathbb{N}$ with $n \geq 2$. Then, the following inequality holds:*

$$\left| \frac{7}{45} A\left(\kappa^n, \aleph^n\right) + \frac{16}{45} A^n\left(\kappa, \kappa, \kappa, \aleph\right) + \frac{2}{15} A^n\left(\kappa, \aleph\right) + \frac{16}{45} A^n\left(\kappa, \aleph, \aleph, \aleph\right) - L_n^n\left(\kappa, \aleph\right) \right|$$
$$\leq \frac{509 n(n-1)(\aleph - \kappa)^2}{273375} \left(\kappa^{n-2} + \aleph^{n-2} \right). \tag{19}$$

Proof. Applying Theorem 2.2 to the function $f(u) = u^n$, we have $f''(u) = n(n-1)u^{n-2}$.

Since $|f''|$ is convex on $[\kappa, \aleph]$, all the assumptions of Theorem 2.2 are satisfied. Furthermore,

$$\frac{1}{\aleph - \kappa} \int_\kappa^\aleph u^n \, du = \frac{\aleph^{n+1} - \kappa^{n+1}}{(n+1)(\aleph - \kappa)} = L_n^n(\kappa, \aleph).$$

Also,

$$f''(\kappa) = n(n-1)\kappa^{n-2}, \qquad f''(\aleph) = n(n-1)\aleph^{n-2}.$$

Substituting these expressions into inequality (15) gives the desired result. □

Remark 3.3. Many other interesting inequalities for means can be obtained by applying the different results to the function $f(u) = \frac{1}{u}$, $u \in \mathbb{R}^+$, however the details are left to the interested reader.

4 Numerical Examples

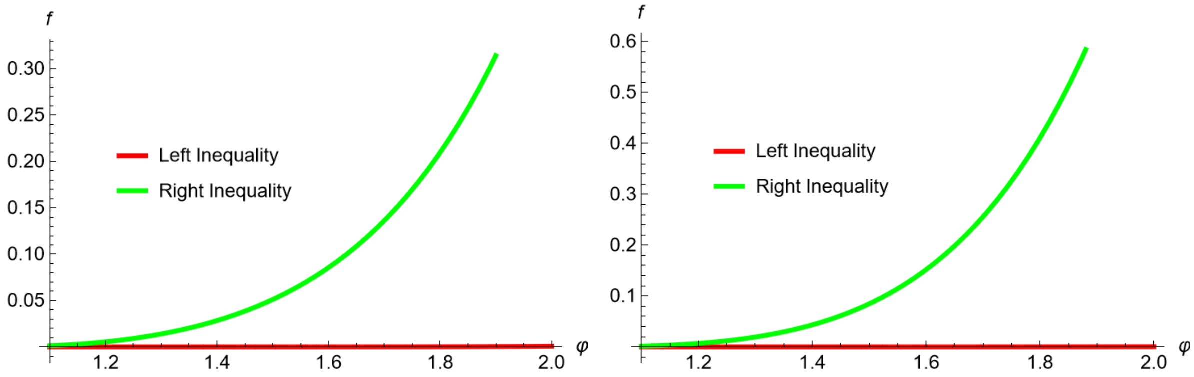
Example 4.1. Let two twice differentiable convex functions $f(\varphi) = e^{2\varphi}$ and $f(\varphi) = \varphi^6$ respectively, in Theorem 2.2, for all $\aleph > 0, \kappa = 1$. Then we examine the subsequent numerical verification (see Tables 1 and 2) and associated graphs (see Figures 1).

φ	Left Inequality	Right Inequality
1.1	2.70155×10^{-10}	0.00122246
1.2	1.91276×10^{-8}	0.00548511
1.3	2.41217×10^{-7}	0.0139774
1.4	1.50157×10^{-6}	0.0284008
1.5	6.35064×10^{-6}	0.05116
1.6	0.0000210389	0.0855866
1.7	0.000058902	0.136315
1.8	0.000145818	0.209665
1.9	0.000328672	0.314239
2.0	0.000688093	0.461659

Table 1: Computed values of the inequalities (15) for $f(\varphi) = e^{2\varphi}$.

φ	Left Inequality	Right Inequality
1.1	3.72024×10^{-10}	0.00137638
1.2	2.38095×10^{-8}	0.00686732
1.3	2.71205×10^{-7}	0.0193852
1.4	1.52381×10^{-6}	0.0432702
1.5	5.81287×10^{-6}	0.0846588
1.6	0.0000173571	0.151893
1.7	0.0000437682	0.255968
1.8	0.0000975238	0.411024
1.9	0.000197709	0.634875
2.0	0.000372024	0.949575

Table 2: Computed values of the inequalities (15) for $f(\varphi) = \varphi^6$.



(a) 2D-plot, for $f(\varphi) = e^{2\varphi}$

(b) 2D-plot, for $f(\varphi) = \varphi^6$

Figure 1: Visual depiction of the inequalities of Theorem 2.2, and Example 4.1

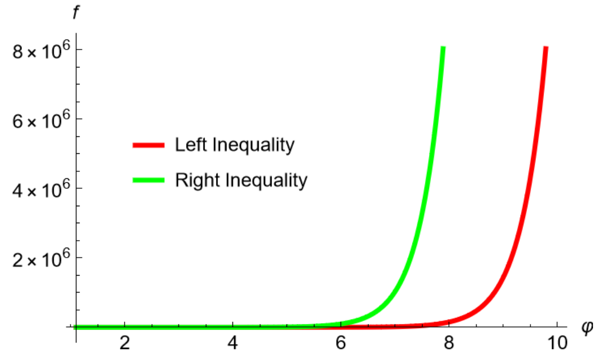
Example 4.2. Let two twice differentiable convex functions $f(\varphi) = e^{2\varphi}$ and $f(\varphi) = \varphi^6$ respectively, in Theorem 2.3, for all $\aleph > 0, q = 2$, and $\kappa = 1$. Then we examine the subsequent numerical verification (see Tables 3 and 4) and associated graphs (see Figures 2).

φ	Left Inequality	Right Inequality
1.1	2.7015×10^{-10}	0.00254
1.2	1.91276×10^{-8}	0.0118397
1.3	2.41217×10^{-7}	0.0314363
1.4	1.50157×10^{-6}	0.0665478
1.5	6.35064×10^{-6}	0.124689
1.6	0.0000210389	0.216457
1.7	0.000058902	0.356571
1.8	0.000145818	0.565239
1.9	0.000328672	0.86997
2.0	0.000688093	1.30794

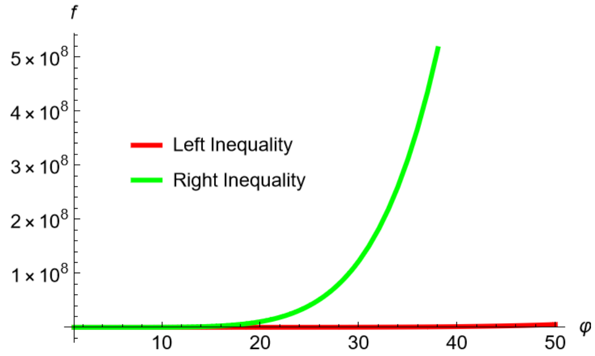
Table 3: Computed values of the inequalities (17) for $f(\varphi) = e^{2\varphi}$.

φ	Left Inequality	Right Inequality
1.1	3.72024×10^{-10}	0.00296
1.2	2.38095×10^{-8}	0.0158615
1.3	2.71205×10^{-7}	0.0476969
1.4	1.52381×10^{-6}	0.112197
1.5	5.81287×10^{-6}	0.228936
1.6	0.0000173571	0.424588
1.7	0.0000437682	0.734325
1.8	0.0000975238	1.20337
1.9	0.00019771	1.8887
2.0	0.000372024	2.86083

Table 4: Computed values of the inequalities (17) for $f(\varphi) = \varphi^6$.



(a) 2D-plot, for $f(\varphi) = e^{2\varphi}$



(b) 2D-plot, for $f(\varphi) = \varphi^6$

Figure 2: Visual depiction of the inequalities of Theorem 2.3, and Example 4.2

Example 4.3. Let two twice differentiable convex functions $f(\varphi) = e^{2\varphi}$ and $f(\varphi) = \varphi^6$ respectively, in Theorem 2.4, for all $\aleph > 0$, $p = q = 2$, and $\kappa = 1$. Then we examine the subsequent numerical verification (see Tables 5 and 6) and associated graphs (see Figures 3).

φ	Left Inequality	Right Inequality
1.1	2.7015×10^{-10}	0.00148
1.2	1.91276×10^{-8}	0.00666332
1.3	2.41217×10^{-7}	0.0171101
1.4	1.50157×10^{-6}	0.0351601
1.5	6.35064×10^{-6}	0.0641966
1.6	0.0000210389	0.108999
1.7	0.000058902	0.176208
1.8	0.000145818	0.27494
1.9	0.000328672	0.4176
2.0	0.000688093	0.620955

Table 5: Computed values of the inequalities (18) for $f(\varphi) = e^{2\varphi}$.

φ	Left Inequality	Right Inequality
1.1	3.72024×10^{-10}	0.00167122
1.2	2.38095×10^{-8}	0.00846512
1.3	2.71205×10^{-7}	0.0244145
1.4	1.52381×10^{-6}	0.0557104
1.5	5.81287×10^{-6}	0.111183
1.6	0.0000173571	0.202863
1.7	0.0000437682	0.34661
1.8	0.0000975238	0.562818
1.9	0.000197709	0.877173
2.0	0.000372024	1.32148

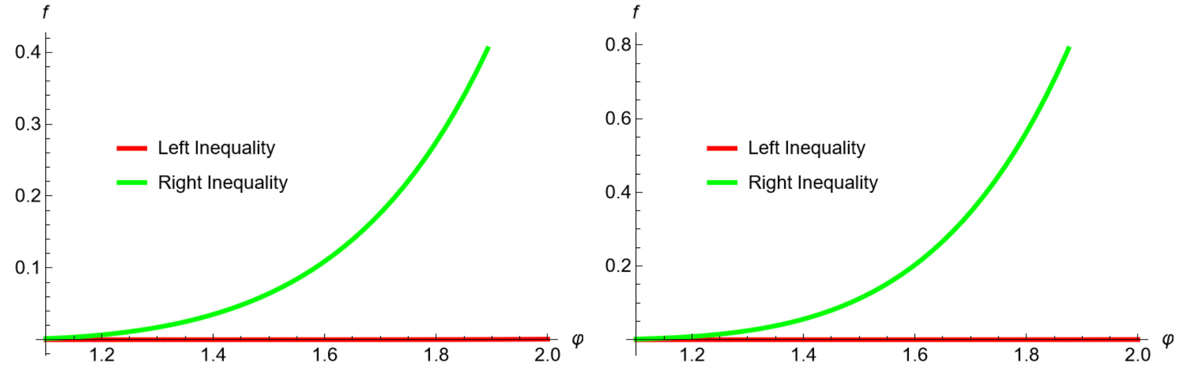
Table 6: Computed values of the inequalities (18) for $f(\varphi) = \varphi^6$.

Quadrature Formulae	Absolute Error
Trapezoidal Formula	14.3571
Simpson 1/3 Rule	0.284226
Simpson 3/8 Rule	0.12669
Boole's Rule	0.000372024

Table 7: Comparison of Absolute Error, taking $f(\varphi) = \varphi^6$, in quadrature formulae.

4.1 Discussion

In this subsection, we give a graphical analysis of the newly derived inequalities to justify the results arrived at. These graphs, along with the corresponding numerical outcomes, give a very clear insight into how derived inequalities are satisfied in the respective cases. From Example 4.1, the corresponding Figure 1 and Tables 1 and 2 depicts the behavior of the inequalities described in Theorem 2.2. The graphical plots show that the inequalities are satisfied for the



(a) 2D-plot, for $f(\varphi) = e^{2\varphi}$ (b) 2D-plot, for $f(\varphi) = \varphi^6$

Figure 3: Visual depiction of the inequalities of Theorem 2.4, and Example 4.2

chosen parameters, and the numerical data in the tables further confirm their validity. The close agreement between the theoretical bounds and the computed values underscores the accuracy of the results.

Following the same pattern for Theorem 2.3 and 2.4, one can examine Example 4.2 and Example 4.3 respectively. By doing so, we can sufficiently support the inequalities presented in the Theorems. The obtained graphical and numerical values are also consistent with the theoretical values, which makes the work very confident about the new results derived in the present work. The use of graphical representation alongside enforced numerical validation provides the most inclusive impact assessment on the behavior of the obtained inequalities.

Furthermore, error can be reduced using a suitable technique based on the polynomial's degree. We use $f(\varphi) = \varphi^6$ for Boole's, Simpson's 1/3, Simpson's 3/8, and trapezoidal formula, for instance, as shown in Table 7. A higher-degree polynomial has a very large error when solved using a trapezoidal formula but a very small error when solved using Boole's rule. Consequently, if we implement the right technique according to the degree of the polynomial, errors can be reduced.

5 Conclusion

In conclusion, this article significantly advances the understanding of Boole's type inequalities, particularly in the context of twice-differentiable convex functions. This research helps to improve the error bounds of Boole's formula in numerical integration. Initially, the new identity is formulated involving twice differentiable functions. Utilizing this equality, numerous significant inequalities are established by employing convexity, Hölder's inequality, and the power-mean inequality. Numerical examples and graphical illustrations are incorporated to substantiate the validity and importance of these recently established inequalities. In the future, these results can be extended by using different convexities and other forms of calculus. Researchers can explore further extensions and delve deeper into the implications of these findings across other mathematical domains.

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Declarations

Conflict of interest

The authors declare that they have no conflict of interest.

Authors' contributions

All authors contributed equally to this work.

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