Characterization of Positive Operators[*]

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Abstract. *A characterization of positive operators on finite dimensional complex vector spaces, developed from the Routh-Hurwitz criterion with review of some basic concepts and results.*
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1. Introduction

Here we give a characterization of the positive operators within the class of self-adjoint operators on finite dimensional complex vector spaces. The result is a direct consequence of the *Routh-Hurwitz theorem* [1] which establishes a necessary and sufficient condition for all eigenvalues of a real polynomial to have negative real part.

There are several results concerning positive operators, like a necessary condition given by a theorem of Perron generalized by Frobenius [2, pp.64–65] (and related results as in [3]) and a sufficient condition given in [4]. However, our theorem provides a necessary and sufficient condition that can be effectively verified without explicit calculation of eigenvalues. This procedure can be reduced to an algorithm which allows computational implementation.

This article has the following structure. Section 2 reviews basic facts about positive operators and characteristic polynomials; Section 3 presents the *Routh-Hurwitz criterion* and extensions that allows one determine whether all (complex) roots of a real polynomial have non-negative real part; our main theorem is stated in Section 4 and proved from earlier results; finally, Section 5 presents the application of the main theorem in two and three dimensions.

2. Positive operators

Let $\mathcal{H}$ be a Hilbert space and $\langle , \rangle$ its inner product (anti-linear in the first entry and linear in the second entry). The norm associated with this inner product is given by:

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}, \ \forall \varphi \in \mathcal{H}.$$

An *operator in* $\mathcal{H}$ means a bounded linear function from $\mathcal{H}$ to $\mathcal{H}$. I denotes the identity operator and $\ker(A)$ the kernel of an operator $A : \mathcal{H} \to \mathcal{H}$:

$$\ker(A) = \{\varphi \in \mathcal{H};\ A\varphi = 0\}.$$

For concepts and results not explicitly provided, we use notations and definitions as given in references [5], [6] and [7].

Definition 2.1. The adjoint operator of and operator $A : \mathcal{H} \to \mathcal{H}$ is the operator $A^* : \mathcal{H} \to \mathcal{H}$ satisfying:[1]

$$\langle A^*\varphi, \psi \rangle = \langle \varphi, A\psi \rangle,\ \forall \varphi, \psi \in \mathcal{H}.$$

A is self-adjoint when $A^* = A$, *i.e.*,

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle,\ \forall \varphi, \psi \in \mathcal{H}.$$

A is positive when

$$\langle \varphi, A\varphi \rangle \geq 0,\ \forall \varphi \in \mathcal{H}.$$

Proposition 2.2. *Any positive operator is self-adjoint also.*

This proposition follows from *polarization identity* [5, p.171-172] [7, p.4], which provides the inner product in terms of the norm:

$$\langle \psi, \varphi \rangle = \frac{1}{4}\left\{ \left(\|\psi + \varphi\|^2 - \|\psi - \varphi\|^2\right) - i\left(\|\psi + i\varphi\|^2 - \|\psi - i\varphi\|^2\right) \right\},\ \forall \psi, \varphi \in \mathcal{H}.$$

Our goal is to find a criterion for characterizing positive operators within the set of self-adjoint operators. The argument of Section 4 uses some definitions and results of *Spectral Theory*, the relevant facts of which are recalled below.

Definition 2.3. An *eigenvalue* of an operator A is a number $\lambda \in \mathbb{C}$ such that $\ker(\lambda I - A) \neq \{0\}$; in this case, the elements of $\ker(\lambda I - A) \setminus \{0\}$ are called *eigenvectors of A associated with the eigenvalue* λ. Explicitly, $\varphi \in \mathcal{H}$ is an eigenvector of A associated with the eigenvalue $\lambda \in \mathbb{C}$ if, and only if,

$$\varphi \neq 0\ \text{ and }\ A\varphi = \lambda\varphi.$$

If $\mathcal{H}$ has finite dimension, the *spectrum* of A is the set of its eigenvalues:

$$\sigma_A := \{\text{eigenvalues of } A\}.$$

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[1] The existence of the adjoint operator follows from *Riesz representation theorem* [6, p.39] [7, p.12, p.31].

When $\mathcal{H}$ has finite dimension, the *characteristic polynomial* of A is defined by

$$p_A(z) := \det(zI - A). \tag{1}$$

In this situation, the spectrum of A is exactly the set of roots of the characteristic polynomial of A:

$$\sigma_A = \{z \in \mathbb{C} : p_A(z) = 0\}.$$

Proposition 2.4. *Let A be an operator in the Hilbert space $\mathcal{H}$.*

　　i) If A is self-adjoint, then its eigenvalues are real numbers;

　　ii) If A is positive, then its eigenvalues are non-negative real numbers.

Proof. Let λ be an eigenvalue of A and let $\varphi \in \mathcal{H}$ ($\varphi \neq 0$) be an eigenvector of A associated with λ. If A is self-adjoint, then

$$\lambda = \frac{\langle \varphi, \lambda\varphi \rangle}{\|\varphi\|^2} = \frac{\langle \varphi, A\varphi \rangle}{\|\varphi\|^2} = \frac{\langle A\varphi, \varphi \rangle}{\|\varphi\|^2} = \frac{\langle \lambda\varphi, \varphi \rangle}{\|\varphi\|^2} = \bar{\lambda};$$

so, $\lambda \in \mathbb{R}$. If A is positive, then

$$\lambda = \frac{\langle \varphi, \lambda\varphi \rangle}{\|\varphi\|^2} = \frac{\langle \varphi, A\varphi \rangle}{\|\varphi\|^2} \geq 0.$$

$\square$

Theorem 2.5 (Spectral). *Let $\mathcal{H}$ be separable and let A be a (bounded) operator in $\mathcal{H}$. Then, A is self-adjoint if, and only if, there exists a sequence of orthogonal projections $(P_1, P_2, ...)$, viz.,*

$$P_k P_j = P_k P_j = \delta_{kj} P_k, \ \forall k \neq j = 1, 2, ...,$$

and a sequence of real numbers $(\lambda_1, \lambda_2, ...)$ such that[2]

$$I = \sum_k P_k, \ A = \sum_k \lambda_k P_k.$$

In this case, it holds

$$\sigma_A = \{\lambda_1, \lambda_2, ...\}$$

and

$$\mathrm{Im}(P_k) = \ker(\lambda_k I - A), \ \forall k = 1, 2, ...$$

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[2]When the dimension of $\mathcal{H}$ is infinite, the sum can be a convergent serie (w.r.t. the norm topology).

Proposition 2.6. *Let $\mathcal{H}$ be separable and let A be a self-adjoint operator in $\mathcal{H}$. Then, A is positive if, and only if, its eigenvalues are non-negatives.*

Proof. The implication was already demonstrated in Proposition 2.4; so, here we prove only the reciprocal sentence. For this, consider the spectral decomposition of A according to Theorem 2.5; then, for all $\varphi \in \mathcal{H}$, it holds

$$\langle \varphi, A\varphi \rangle = \left\langle \varphi, \sum_k \lambda_k P_k \varphi \right\rangle = \sum_k \lambda_k \langle \varphi, P_k \varphi \rangle = \sum_k \lambda_k \langle P_k \varphi, P_k \varphi \rangle = \sum_k \lambda_k \underbrace{\|P_k \varphi\|^2}_{\geq 0}.$$

Therefore, if all eigenvalues of A are non-negatives, the last sum above is non-negative also. From this, it follows that that A is positive, according to Definition 2.1. $\square$

Proposition 2.7. *Assume that $\mathcal{H}$ is finite dimensional and let A be a self-adjoint operator on $\mathcal{H}$. Then, the characteristic polynomial of A is real (i.e., it has real coefficients).*

Proof. Let $\bar{p}_A(z)$ be the polynomial whose coefficients are the corresponding complex conjugated of the coefficients of the characteristic polynomial $p_A(z)$; then

$$\bar{p}_A(z) = \overline{p_A(\bar{z})}, \ \forall z \in \mathbb{C}.$$

Using some algebraic properties of the conjugation $*$ (which assigns each operator to its adjoint), it follows that

$$\overline{p_A(z)} = \overline{\det(zI - A)} = \det(zI - A)^* = \det(\bar{z}I - A) = p_A(\bar{z}), \ \forall z \in \mathbb{C}.$$

Combining the two identities above, it follows:

$$\bar{p}_A(x) = p_A(x), \ \forall x \in \mathbb{R}.$$

Therefore, the polynomials $\bar{p}_A(z)$ and $p_A(z)$ have the same restriction to $\mathbb{R}$, what implies that they have the same coefficients and from what follows that the coefficients of $p_A(z)$ are real numbers. $\square$

From Theorem 2.6 and Proposition 2.7, we realize that one can obtain a characterization of the positive operators within the class of self-adjoint operators in $\mathbb{C}^n$ using any characterization of the real polynomials whose complex roots have non-negative real part. This points to the *Routh-Hurwitz criterion*.

3. Routh-Hurwitz criterion and extensions

Definition 3.1. Let $p(z)$ be a polynomial with complex coefficients:

$$p(z) = b_0 z^n + b_1 z^{n-1} + ... + b_{n-1} z + b_n.$$

The *Hurwitz determinants* of $p(z)$ are defined by

$$\Delta_k := \det \begin{pmatrix} b_1 & b_3 & b_5 & b_7 & \cdots & \cdots & b_{2k-1} \\ b_0 & b_2 & b_4 & b_6 & \cdots & \cdots & b_{2k-2} \\ 0 & b_1 & b_3 & b_5 & \cdots & \cdots & b_{2k-3} \\ 0 & b_0 & b_2 & b_4 & \cdots & \cdots & \vdots \\ \vdots & 0 & b_1 & b_3 & b_5 & \cdots & b_{k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & b_{k+1} \\ 0 & 0 & \cdots & \cdots & b_{k-4} & b_{k-2} & b_k \end{pmatrix} \quad (k = 1, 2, ..., n),$$

where

$$b_j = 0, \ \forall j \in \mathbb{N}, \ j > n.$$

In particular, for $n = 3$ this definition means

$$\Delta_1 = b_1, \quad \Delta_2 = \det \begin{pmatrix} b_1 & b_3 \\ b_0 & b_2 \end{pmatrix}, \quad \Delta_3 = \det \begin{pmatrix} b_1 & b_3 & 0 \\ b_0 & b_2 & 0 \\ 0 & b_1 & b_3 \end{pmatrix}.$$

Theorem 3.2 (Routh-Hurwitz criterion). *Consider the polynomial (3.1) has real coefficients and that its leading coefficient is positive ($b_0 > 0$). Then, the roots of this polynomial have negative real part if, and only if, the Hurwitz determinants of the polynomial are positive:*

$$\Delta_k > 0, \ \forall k = 1, ..., n.$$

We refer to [2, p.231] for a proof of this theorem.

In what follows, we state and demonstrate theorems analogous to the Routh-Hurwitz criterion. By convention, the phrase "$w \in \mathbb{C}$ is a root of the polynomial $p(z)$ with multiplicity zero" means that "w is not a root of $p(z)$".

Proposition 3.3 (Extended Routh-Hurwitz criterion). *Consider a polynomial $p(z)$ having real coefficients and positive leading coefficient. Let n_0 be the multiplicity of number zero as a root of $p(z)$. Then, the non-zero roots of $p(z)$ have negative real part if, and only if,*

its Hurwitz determinants with indices from 1 to $n - n_0$ are positive:

$$\Delta_k > 0, \ \forall k = 1, ..., n - n_0.$$

Proof. It follows from n_0's definition that $0 \leq n_0 < n$. Now, let $p(z)$ be as in Eq.3.1. The case $n_0 = 0$ reduces to the Routh-Hurwitz criterion (Theorem 3.2); therefore, we assume that $n_0 \geq 1$. From the hypothesis, the coefficients of the monomials of $p(z)$ having degree less than n_0 are equal to zero

$$b_{n-n_0+1} = ... = b_n = 0. \tag{2}$$

Consider the following auxiliary polynomial

$$\hat{p}(z) := b_0 z^{n-n_0} + b_1 z^{n-n_0-1} + ... + b_{n-n_0-1}z + b_{n-n_0}.$$

Therefore

$$p(z) = z^{n_0} \hat{p}(z).$$

The polynomial $\hat{p}(z)$ has degree $n - n_0$ and its roots are equal to the non-zero roots of $p(z)$; further, the Hurwitz determinants of $\hat{p}(z)$ and $p(z)$ are also equal due to Eq.2 and Definition 3.1. As happens with $p(z)$, $\hat{p}(z)$ is a real polynomial with positive leading coefficient; therefore, the thesis follows from Theorem 3.2 applied to $\hat{p}(z)$. $\square$

Remark 3.4. In the case $p(z) = z^n$ (the number zero is a root of the polynomial with multiplicity equal to the polynomial's degree), all Hurwitz determinants of $p(z)$ are zero and (evidently) the polynomial has no non-zero root.

Proposition 3.5 (Symmetric of the extended Routh-Hurwitz criterion)**.** *Consider a polynomial $p(z)$ having real coefficients and positive leading coefficient. Let n_0 be the multiplicity of the number zero as a root of $p(z)$. Then, the non-zero roots of $p(z)$ have positive real part if, and only if, the Hurwitz determinants satisfy the condition:*

$$(-1)^{1+\lfloor k/2 \rfloor} \Delta_k > 0, \ \ \forall k = 1, 2, ..., n - n_0, \tag{3}$$

where $\lfloor k/2 \rfloor$ is the greatest integer less than or equal to $k/2$:

$$\lfloor x \rfloor := \max \{ m \in \mathbb{Z}; \ m \leq x \} \ \ (x \in \mathbb{R}).$$

Therefore, the conditions given by Eq.3 are necessary and sufficient to guarantee that all roots of the polynomial $p(z)$ have non-negative real part.

Proof. It follows from n_0's definition that $0 \leq n_0 < n$. Now, consider the auxiliary

polynomial

$$\tilde{p}(z) := (-1)^n p(-z).$$

Then, the non-zero roots of $p(z)$ have positive real part if, and only if, all non-zero roots of $\tilde{p}(z)$ have negative real part. The coefficients of $\tilde{p}(z)$ are given in terms of the coefficients of $p(z)$ by

$$\tilde{b}_k = (-1)^k b_k \ \ \forall k = 1, 2, ..., n.$$

Therefore, the Hurwitz determinants of $p(z)$ and $\tilde{p}(z)$ satisfy the following identities:[3]

$$\tilde{\Delta}_k = (-1)^{1+\lfloor k/2 \rfloor} \Delta_k \ \ \forall k = 1, 2, ..., n.$$

As with $p(z)$, $\tilde{p}(z)$ is a real polynomial with positive leading coefficient ($\tilde{b}_0 = b_0$) and the number zero is a root of it with multiplicity n_0; therefore, the thesis follows from Proposition 3.3 applied to $\tilde{p}(z)$. □

4. Characterization of positive operators

Theorem 4.1. *Let A be a self-adjoint operator in $\mathbb{C}^n$ and let $n_0 \in \{0, 1, ..., n\}$ be the multiplicity of the number zero as a root of the characteristic polynomial of A. Then, A is a positive operator if, and only if,*

$$(-1)^{1+\lfloor k/2 \rfloor} \Delta_k > 0 \ \ \forall k = 1, 2, ..., n - n_0.$$

Proof. The result follows directly from the combination of Proposition 2.6 and Proposition 3.5. □

Naturally, this result motivates us to know expressions for the coefficients of the characteristic polynomial of operators. So, we highlight the following results:

Proposition 4.2. *Let A be a self-adjoint operator in $\mathbb{C}^n$ and let its characteristic polynomial be*

$$p_A(z) = \det(zI - A) =: b_0 z^n + b_1 z^{n-1} + ... + b_{n-1} z + b_n.$$

i) Recursive formula in terms of traces:

$$b_0 = 1; \ b_k = -\frac{1}{k} \left\{ \sum_{j=1}^k b_{k-j} \mathrm{tr}\left(A^j\right) \right\}, \ \ \forall k = 1, ..., n.$$

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[3]The signal cames from the number $1 + \lfloor k/2 \rfloor$ of lines of the matrix defining $\tilde{\Delta}_k$ which are different by a factor -1 from the corresponding lines of the matrix defining Δ_k: by a basic property of the determinant, each different line implies a factor -1 in the relation between the two determinants.

ii) Recursive formula in terms of subdeterminants:

$$b_0 = 1; \; b_k = (-1)^k \sum_{j_1 < ... < j_k} \det \begin{pmatrix} a_{j_1 j_1} & a_{j_1 j_2} & \cdots & a_{j_1 j_k} \\ a_{j_2 j_1} & a_{j_2 j_2} & \cdots & a_{j_2 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j_k j_1} & a_{j_k j_2} & \cdots & a_{j_k j_k} \end{pmatrix}, \; \forall k = 1, ..., n.$$

In particular,

$$b_1 = -\operatorname{tr}(A), \; b_n = (-1)^n \det(A).$$

Proofs for such formulas can be found in [8] and [9], respectively. A different formula for the coefficients of the characteristic polynomial of a self-adjoint operator can be found in [10].

Finally, we collect the facts in a simple algorithm:

Algorithm for Characterization of positive operators

Let A be a operator in $\mathbb{C}^n$ and consider its representation in the canonical basis (or any other orthonormal basis) of $\mathbb{C}^n$:

$$[A] = (a_{ij})_{i,j=1,...,n}.$$

1. Check that the operator A is self-adjoint, *i.e.*, if $\bar{a}_{ij} = a_{ji} \; \forall i, j = 1, ..., n$;

2. Calculate the coefficients of the characteristic polynomial of A, using one of the formulas in Proposition 4.2;

3. Calculate the multiplicity of the number zero as a root of the characteristic polynomial of A, using, for example, the formula

$$n_0 = \min \left\{ k \in \mathbb{N} \; / \; \left. \frac{d^k p_A}{dz^k} \right|_{z=0} \neq 0 \right\}.$$

4. Calculate the Hurwitz determinants of the characteristic polynomial of A up to the order $n - n_0$, using Definition 3.1;

5. Check if the calculated Hurwitz determinants satisfy the condition of Theorem 4.1.

The computational efficiency of this Algorithm 4 could be compared with the computational efficiency to calculate the operator's eigenvalues with sufficient precision. We let this for the reader.

5. Special Cases

In this section we get from Theorem 4.1 explicit conditions for a self-adjoint operator to be positive on two and three dimensions. These expressions are presented in terms of determinants and traces, following Definition 3.1 and Proposition 4.2.

5.1. Dimension $n = 2$

Let A be a self-adjoint operator in $\mathbb{C}^2$ and consider its matrix representation with respect to the canonical basis (or with respect to any other orthogonal basis):

$$[A] = (a_{ij})_{i,j=1,2}\,.$$

Self-adjointness means

$$a_{ij} = \bar{a}_{ji} \in \mathbb{C} \ \ \forall i,j = 1,2.$$

The coefficients of the characteristic polynomial of A are, as one can obtain directly from Eq.(1) or from Theorem 4.2:

$$b_0 = 1\,,\ b_1 = -\operatorname{tr}(A)\,,\ b_2 = \det(A)\,.$$

The Hurwitz determinants are:

$$\Delta_1 = b_1 = -\operatorname{tr}(A)\,,\ \Delta_2 = b_1 b_3 - b_0 b_2 = \det(A)\,.$$

Now, to get the necessary and sufficient conditions for A be positive, we have to consider three different situations, distinguished by the multiplicity, denoted here by μ, of the number zero as a root of the characteristic polynomial of A.

<u>Case $\mu = 0$</u> ($b_2 = \det(A) \neq 0$): Theorem 4.1 stablishes that A is positive if, and only if,

$$\operatorname{tr}(A) > 0,\ \det(A) > 0.$$

<u>Case $\mu = 1$</u> ($b_2 = \det(A) = 0\,,\ b_1 = -\operatorname{tr}(A) \neq 0$): Theorem 4.1 stablishes that A is positive if, and only if,

$$\operatorname{tr}(A) > 0.$$

<u>Case $\mu = 2$</u> ($b_2 = \det(A) = 0\,,\ b_1 = -\operatorname{tr}(A) = 0$): Theorem 4.1 stablishes that A is positive (without any further condition).

Finally, we can collect all two dimensional cases in a single sentence:

A self-adjoint operator A in $\mathbb{C}^2$ is positive if, and only if, it satisfies one out of the two following conditions:

$$\begin{cases} (i) \ \det(A) = 0 \,, \ \operatorname{tr}(A) \geq 0; \\ (ii) \ \det(A) > 0 \,, \ \operatorname{tr}(A) > 0. \end{cases}$$

5.2. Dimension $n = 3$

Let A be a self-adjoint operator in $\mathbb{C}^3$ and consider its matrix representation with respect to the canonical basis (or with respect to any other orthonormal basis):

$$[A] = (a_{ij})_{i,j=1,2,3} \,.$$

As above, self-adjointness means

$$a_{ij} = \bar{a}_{ji} \in \mathbb{C} \ \ \forall i,j = 1,2,3.$$

The coefficients of the characteristic polynomial of A are, as one can obtain directly from (1) or from Theorem 4.2:

$$\begin{aligned} b_0 &= 1; \\ b_1 &= -\operatorname{tr}(A); \\ b_2 &= -\frac{1}{2}\left(b_1\operatorname{tr}(A) + b_0\operatorname{tr}\left(A^2\right)\right) = \frac{1}{2}\left(\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)\right); \\ b_3 &= -\frac{1}{3}\left\{b_2\operatorname{tr}(A) + b_1\operatorname{tr}\left(A^2\right) + b_0\operatorname{tr}\left(A^3\right)\right\} = -\det(A). \end{aligned}$$

The Hurwitz determinants of the characteristic polynomial are:

$$\Delta_1 = b_1 = -\operatorname{tr}(A),$$

$$\Delta_2 = b_1 b_2 - b_0 b_3 = -\frac{1}{2}\operatorname{tr}(A)\left(\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)\right) + \det(A),$$

$$\Delta_3 = b_1 b_2 b_3 - b_0 b_3^2 = \left\{\frac{1}{2}\operatorname{tr}(A)\left(\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)\right) - \det(A)\right\}\det(A).$$

Now, to get the necessary and sufficient conditions for A be positive, we have to consider four different situations, distinguished by the multiplicity of the number zero as a root of the characteristic polynomial of A. For the sake of space and to avoid boredom, we consider only the case in which the number zero is not a root of the characteristic polynomial ($b_3 = -\det(A) \neq 0$). In this case, Theorem 4.1 stablishes that A is positive, if and only

if,

$$\begin{cases} \operatorname{tr}(A) > 0, \\ \dfrac{1}{2}\operatorname{tr}(A)\left(\operatorname{tr}(A)^2 + \operatorname{tr}(A^2)\right) + \det(A) > 0, \\ \left(\dfrac{1}{2}\operatorname{tr}(A)\left(\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)\right) - \det(A)\right)\det(A) > 0. \end{cases}$$

References

[1] Hurwitz A. Uber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt. Mat Ann. 1895; 46: 273-84.

[2] Gantmacher FR. Applications of the Theory of Matrices. Interscience Publishers Inc.; 1959.

[3] Johnson CR, Tarazaga P. A Characterization of Positive Matrices. Positivity. 2005; 9: 149-50.

[4] Fong CK, Tsui SK. A Note on Positive Operators. J Operator Theory. 1981; 5: 73-6.

[5] Halmos PR. Finite Dimensional Vector Spaces. Springer-Verlag; 1997.

[6] Halmos PR. Introduction to Hilbert Space and the Theory of Spectral Multiplicity. Chelsea Publiching Company; 1951.

[7] Conway JB. A Course in Functional Analysis. Springer-Verlag; 1985.

[8] Lewin M. On the coefficients of the characteristic polynomial of a matrix. Discrete Mathematics. 1994; 125: 255-62.

[9] Collings BJ. Characteristic polynomial by diagonal expansion. The American Statistician. 1987; 37(3): 233-5.

[10] Pennisi LL. Coefficients of the Characteristic Polynomial. Mathematics Magazine. 1987; 60: 31-3.