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An overview on real division algebras*

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Abstract. *In this paper, we expose initial concepts of real division algebras, providing historical notes on* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$ *and* $\mathbb{O}$*. This is done to emphasize the relevance of topological K-theory through the Bott-Milnor-Kervaire Theorem, which is at the end. For completeness, we also present some classical results about the main algebras.*
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2. Introduction

Algebraic K-theory originated in the late 1950s as a generalization by *Alexander Grothendieck* (1928-2014) of the famous Riemann-Roch Theorem. Roughly speaking, Grothendieck associated a group $K(X)$ with each X from some family of algebraic spaces, thence recovering the classical Riemann-Roch Theorem as a special case of a result involving K-groups.

Afterwards, *Friedrich Hirzebruch* (1927-2012) and *Michael Atiyah* (1929-2019) realized that Grothendieck's ideas could be applied to the world of Algebraic Topology. The resulting K-theory of topological spaces, referred to as topological K-theory, proved to be quite powerful. For example, some of its initial achievements include determining the maximum number of linearly independent vector fields on spheres, a classification theorem for real division algebras, and the Atiyah-Singer Index Theorem.

In this paper, we focus on the classification theorem for real division algebras. The beauty of this topic becomes apparent to the reader, as it consists in the fact that the techniques of topological K-theory only come into play at the end. Until the last section,

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we will establish the problem with many algebraic details, and then prove it following [1], [4] and [5].

More precisely, in Section 3, we introduce the first definitions in the setting of division algebras. In Section 4, we present key examples of division algebras, providing historical notes that indicate the long-term importance of the problem of division algebras. In Section 5, we study star-algebras to introduce the Cayley-Dickson construction, which serves as motivation for the main theorem under consideration. In Section 6, we present the Bott-Milnor-Kervaire Theorem and some complementary results that play interesting roles in the current discussion. Finally, in Section 7, we provide an overview of the proof for the desired theorem using notions of topological K-theory.

3. First definitions

We begin this section with the elementary notion of real (division) algebra. Thence, we fix some terminology, establish morphisms between algebras and provide elementary facts.

Definition 3.1. Let $\mathcal{A}$ be a finite-dimensional real vector space and $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a bilinear map. The pair $(\mathcal{A}, m)$ is a:

- **real algebra** provided that there exists a non-zero element $1 \in \mathcal{A}$ such that $m(1, a) = m(a, 1) = a$ for all $a \in \mathcal{A}$;
- **real division algebra** provided that it is a real algebra in which there are no zero divisors. This means that, if $a, b \in \mathcal{A}$ are such that $m(a, b) = 0$, then either $a = 0$ or $b = 0$. $\diamondsuit$

We say that m as above is a **multiplication** in $\mathcal{A}$. We also say that $\mathcal{A}$ is a real (division) algebra, omitting its multiplication, and we write ab instead of $m(a, b)$ for all $a, b \in \mathcal{A}$.

Definition 3.2. A real algebra $\mathcal{A}$:

- is **commutative** if $ab = ba$ for all $a, b \in \mathcal{A}$;
- is **associative** if $(ab)c = a(bc)$ for all $a, b, c \in \mathcal{A}$;
- is **alternative** if $a^2b = a(ab)$ and $ab^2 = (ab)b$ for all $a, b \in \mathcal{A}$, which is the same, due to *Emil Artin* (1898-1962), as every subalgebra of $\mathcal{A}$ generated by two elements being associative;
- is **normed** if it is equipped with a norm $|\cdot| : \mathcal{A} \to [0, \infty)$ such that $|a|\,|b| = |ab|$ for all $a, b \in \mathcal{A}$;
- has **multiplicative inverses** if there exits $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = a^{-1}a = 1$ for every non-zero $a \in \mathcal{A}$. $\diamondsuit$

The term “alternative” comes from the fact that the *associator*

$$\begin{aligned} [\cdot,\cdot,\cdot]: \mathcal{A}\times\mathcal{A}\times\mathcal{A} &\rightarrow \mathcal{A} \\ (a,b,c) &\mapsto (ab)c-a(bc) \end{aligned}$$

alternates in an alternative algebra, that is, the associator changes sign under an odd permutation of the letters a, b and c, but remains unchanged under an even permutation. At this point, the reader may have noted a parallel between the *associator* and the *commutator* $[\cdot,\cdot]:\mathcal{A}\times\mathcal{A}\rightarrow\mathcal{A}$ given by $[a,b]=ab-ba$, which is identically zero in a commutative algebra.

Remark 3.3. The following facts hold true.

- The absence of zero divisors is equivalent to the operations of left and right multiplication by non-zero elements being invertible. This follows from the *Rank-Nullity Theorem*.
- Every associative real algebra is an alternative real algebra. The converse is false, as shown in Remark 4.5.
- In a normed division algebra, we have $|1|=1$ since $|1|^2=|1|\,|1|=|1|$. Moreover, a normed real algebra $\mathcal{A}$ is necessarily a division algebra. Indeed, if $\mathcal{A}$ were not a division algebra, then, for any zero divisors $a,b\in\mathcal{A}$, we would have the absurd $0=|ab|=|a||b|>0$.
- An alternative real algebra with multiplicative inverses is necessarily a real division algebra. This happens because, if $ab=0$ and a is not zero, then $b=(a^{-1}a)b=a^{-1}(ab)=a^{-1}0=0$.
- An alternative and commutative real algebra has multiplicative inverses if and only if it is a division algebra. On the other hand, there exist non-commutative alternative real division algebras without multiplicative inverses. For instance, setting $e_1^2=e_2-1$ in Table 2, we create a real division algebra such that e_1 has no multiplicative inverse, because e_3-e_1 and $-(e_1+e_3)$ are, respectively, left and right inverses for e_1. ♢

Definition 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be real algebras. A linear map $\varphi:\mathcal{A}\rightarrow\mathcal{B}$ is:

- a **homomorphism of real algebras** if $\varphi(1)=1$ and $\varphi(ab)=\varphi(a)\varphi(b)$ for all $a,b\in\mathcal{A}$;
- an **anti-homomorphism of real algebras** if $\varphi(1)=1$ and $\varphi(ab)=\varphi(b)\varphi(a)$ for all $a,b\in\mathcal{A}$. ♢

Evidently, an **isomorphism of real algebras** is an invertible homomorphism of algebras and an **anti-isomorphism of real algebras** is an invertible anti-homomorphism of algebras.

Remark 3.5. The following facts hold true.

- The real numbers is a subalgebra of any real algebra $\mathcal{A}$ by means of the injective homomorphism

$$\begin{aligned} \iota : \mathbb{R} &\to \mathcal{A} \\ \alpha &\mapsto \alpha 1. \end{aligned}$$

- if $\mathcal{B}$ is a commutative algebra, then $\varphi : \mathcal{A} \to \mathcal{B}$ is an anti-homomorphism if and only if it is a homomorphism of algebras.
- Let $\mathcal{A}$ and $\mathcal{B}$ be real algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. If φ is surjective, then it is a homomorphism of algebras. Indeed, for all $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$. Therefore,

$$\varphi(1)\, b = \varphi(1)\varphi(a) = \varphi(1a) = \varphi(a) = b.$$

Analogously, $b\,\varphi(1) = b$. From the uniqueness of the multiplicative identity, $\varphi(1) = 1$. A similar assertion happens to be true for anti-homomorphisms of real algebras. ♢

Since a theory without examples is usually left aside, it is time to provide some of them to the reader. This is done in the next section. The examples that are shown there are the most important ones, with applications in so many branches of mathematics that it is impossible to talk about them without saying something about their origins and ramifications.

4. Historical examples

In this section, we present the main division algebras using the language introduced above. After each presentation, we provide some notes on the history of the algebra in question. The expositions are always followed by references in which the reader can find more details.

Example 4.1. The **real division algebra of the real numbers**, denoted by $\mathbb{R}$:

- as a vector space, is the real Euclidean one-dimensional space $\mathbb{R}^1$, whose elements are real multiples of $1 \in \mathbb{R}$;
- as a real division algebra, has the multiplication equal to its scalar product as a vector space. ♢

Historically speaking, it is not an easy task to choose since when the real numbers are in Mathematics. When should one start telling the history of the real numbers? When is it appropriate to start? Is it appropriate to start in:

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- Prehistory with the cavemen and the counting of hunts and provisions?
- Ancient Egypt with the practical problems surrounding the plantings on the Nile margins?
- the discovery of the irrational numbers by the Pythagoreans or even with Eudoxus and his work on incommensurability of quantities?
- the European Middle Ages with the construction of a meaning for negative numbers as independent entities?
- somewhere else in the history of eastern civilizations?

That is not a simple question. In particular, trying to see the real numbers as the historical evolution of the naturals, integers, rationals and irrationals is not coherent with the historical timeline. In fact, for instance, the irrationals appeared centuries before the negative numbers. Thus, the classical pedagogical presentation of the numerical sets play no role in this discussion. Maybe, considering the nowadays stage of development of Mathematics, a plausible and direct answer to that question is the first formalization of the real numbers. Nevertheless, this is another problem: What is the first one? In order to have a clue of the difficulties involved here, the reader can find in [10] more than twenty formalization of the real numbers, which curiously does not begin by *Dedekind's construction* of 1872[1].

Example 4.2. The **real division algebra of the complex numbers**, denoted by $\mathbb{C}$:

- as a vector space, is the real Euclidean two-dimensional space $\mathbb{R}^2$, whose elements are linear combinations of the vectors of its canonical basis $\{1, e_1\}$;
- as a real division algebra, has the multiplication bilinearly induced by the relations in Table 1. $\diamondsuit$

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$
$\mathbf{1}$	1	e_1
$\boldsymbol{e_1}$	e_1	-1

Table 1. This table describes the complex multiplication of the vectors of the canonical basis.

The complex numbers appeared in the context of the problem of explicitly solving a third degree polynomial equation. The mathematicians that are nowadays associated to this kind of equations are *Girolamo Cardano* (1501 - 1576)

[1]From our point of view, *Cantor's construction* of 1873 via equivalence classes of Cauchy sequences of rational numbers is pedagogically interesting. We think so because it not only develops notions that can be applied in further mathematical studies (for instance, the completion of a metric space, useful in *Functional Analysis*), but it also perfects the knowledge of teachers on the irrationals, which is a problematic topic in high school education.

and *Niccolò Fontana*[2] (1500 - 1557). Nonetheless, the first person to solve the cubic equation was *Scipione del Ferro* (1465-1526), who was a professor at the Bologna University. After accomplishing his solution, he trusted the formula to a student of his called *Antonio Maria del Fiore* (XVI-XVII). After some time, Fiore challenged Tartaglia to a mathematical contest, for which Tartaglia rediscovered del Ferro's formula. More than that, Tartaglia won the competition answering all of the problems proposed by del Fiore, while, unfortunately, this one could solve none of the problems suggested by Tartaglia. In turn, Tartaglia told his formula without proof to Cardano, who had sworn to secrecy. With the formula, Cardano deduced a proof. After that, he found out that del Ferro had discovered the formula before Tartaglia. Then, he published it in his book *Ars Magna* (1545). It is important to note that Cardano mentioned del Ferro as the first author and Tartaglia as an independent solver. However, this was not enough to prevent a novel-like contend between Cardano and Tartaglia, which is well documented in the specific literature.

Probably, Cardano introduced the complex numbers in his book Ars Magna. Nevertheless, it is known that *Rafael Bombelli* (1526 - 1572) was responsible for the current notation $\sqrt{-1}$, which he named "più di meno" at the time, while he was studying the application of the Cardano-Tartaglia Formula to the equation $x^3 = 15x + 4$. Other men whose names appear in the history of complex numbers are *Leonhard Euler* (1707-1783), *Jean-Robert Argand* (1768-1822), *Carl Friedrich Gauss* (1777-1855) and *William Rowan Hamilton* (1805-1865). The interested reader can find more details in [7].

Example 4.3. The **real division algebra of the quaternions**, denoted by $\mathbb{H}$:

- as a vector space, is the real Euclidean four-dimensional space $\mathbb{R}^4$, whose elements are linear combinations of the vectors of its canonical basis $\{1, e_1, e_2, e_3\}$;
- as a real division algebra, has the multiplication bilinearly induced by the relations in Table 2, which can be easily deduced from the mnemonic diagram presented in Figure 1. $\diamondsuit$

[2]This one is known as *Tartaglia*, which means "stammerer" in Italian. This nickname is due to serious wounds in his jaw and palate, acquired during a French invasion against Venice, which left him with a speech impediment.

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
$\mathbf{1}$	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1

Table 2. This table describes the quaternionic multiplication of the vectors of the canonical basis.

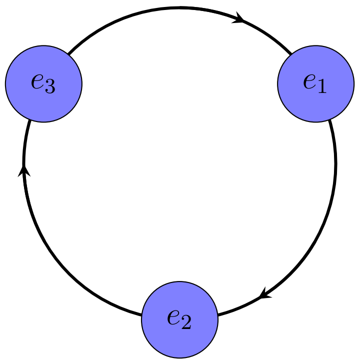

Figure 1. The arrows in this circular diagram indicate the positive sign to obtain the third element from the product of the other ones. For example, $e_3e_1 = e_2$ and $e_1e_2 = e_3$. If we multiply two elements linked by an arrow in the opposite direction, then we have to put a minus sign in front of the third element. For instance, $e_3e_2 = -e_1$ and $e_2e_1 = -e_3$. Moreover, we have to remember that $e_1^2 = e_2^2 = e_3^2 = -1$. This allows us to deduce the equation $e_1e_2e_3 = -1$, which is also an important relation for the quaternions.

William Rowan Hamilton (1805-1865) was the responsible for introducing the quaternions in Mathematics. Interestingly, before developing the quaternions, he was involved with the complex numbers. In 1833, he completed his *Pair Theory*, which was understood at the time as a new algebraic representation for the complex numbers. Nowadays, Hamilton's formulation of the complex numbers is their definition in any first course. In fact, Hamilton wrote a complex number as an ordered pair of real numbers, and defined their sum $(a, b) + (c, d) = (a + c, b + d)$ and their multiplication $(a, b)(c, d) = (ac - bd, ad + bc)$. As a natural step, Hamilton tried to extend the complex numbers to a new algebraic structure in which each element would be composed of one real part and two distinct imaginary parts. This idea would be known as his *Triplets Theory*. Inspired by the way one represents rotations in the plane using complex numbers, Hamilton was carried into this search for his desire to represent rotations in the three-dimensional space in a similar manner. Indeed, much of his work after finding out the quaternions was to publicize them through the idea that they were intrinsically linked with Geometry and Physics.

Nevertheless, Hamilton had failed to create a new algebra for more than ten years, until he found an answer on October 16th, 1843, while he walked with his wife, Lady Hamilton, across the Royal Canal in Dublin, going to a meeting of the Royal Irish Academy. In that moment, he realized that he would need three imaginary parts instead of two. In fact, he noted that the three distinct imaginary parts, which he named i, j and k, should verify the conditions $i^2 = j^2 = k^2 = ijk = -1$. Then, he wrote his results on the stone of the Brougham Bridge, which we unfortunately cannot find today because of the action of time. The reader can find more interesting details in [3] and [9].

Example 4.4. The **real division algebra of the octonions**, denoted by $\mathbb{O}$:

- as a vector space, is the real Euclidean eight-dimensional space $\mathbb{R}^8$, whose elements are linear combinations of the vectors of its canonical basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$;
- as a real division algebra, has the multiplication bilinearly induced by the relations in Table 3, which can be easily deduced from the mnemonic diagram presented in Figure 2. $\diamondsuit$

$\cdot$	$\mathbf{1}$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$
$\mathbf{1}$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
$\boldsymbol{e_2}$	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
$\boldsymbol{e_3}$	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
$\boldsymbol{e_4}$	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
$\boldsymbol{e_5}$	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
$\boldsymbol{e_6}$	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
$\boldsymbol{e_7}$	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 3. This table describes the octonionic multiplication of the vectors of the canonical basis.

The octonions were first described by *John Thomas Graves* (1806 - 1870), who was a Hamilton's friend since both attended together the Trinity College in Dublin. In fact, Graves' interest in algebra was particularly responsible for Hamilton's enterprise on the complex numbers and on the triplets. At the same day of his decisive walk across the Royal Canal, Hamilton sent a letter to Graves describing the quaternions. Graves answered greeting him by the boldness of his idea, adding that "There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties." Moreover, Graves asked "If with your alchemy you can make three pounds of gold, why should you stop there?"

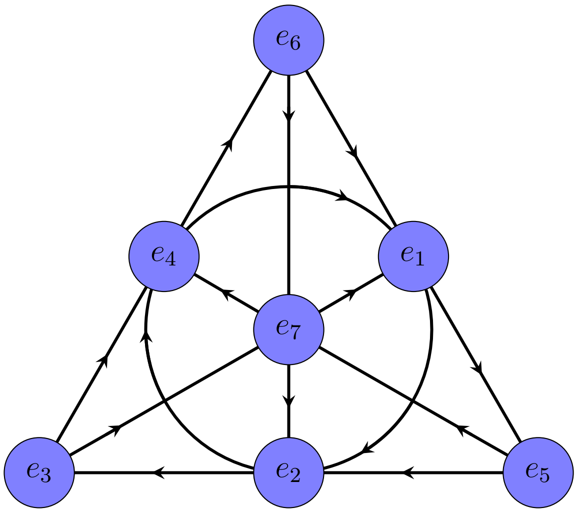

Figure 2. The mathematical object that allowed the construction of this diagram is known as the Fano Plane, which was developed by *Gino Fano* (1871 - 1952). This is the finite projective plane with the least number of points and lines. It has seven points and seven lines, with three points on each line and three lines through each point. We use the arrows in this diagram to indicate the positive sign to obtain the third element of each line from the product of the other ones. For example, $e_4e_6 = e_3$ and $e_7e_2 = e_6$. If we multiply two elements linked by an arrow in the opposite direction, then we have to put a minus sign in front of the third element. For instance, $e_1e_4 = -e_2$ and $e_1e_7 = -e_3$. Further, we have to remember that $e_1^2 = e_2^2 = e_3^2 = e_4^2 = e_5^2 = e_6^2 = e_7^2 = -1$. Note that the expression $e_1e_2e_3e_4e_5e_6e_7$ has no meaning since $\mathbb{O}$ is not associative.

On December 26th, 1843, Graves wrote to Hamilton a description of a new normed division algebra of eight dimensions, which he called *octaves*. On January, 1844, Graves sent three letters to Hamilton expanding his discoveries. He even considered the idea of a *General Theory of* 2^m*-ions* and tried to construct a normed division algebra of sixteen dimensions. On July, 1844, Hamilton answered Graves pinpointing that the octonions were non-associative. Indeed, Hamilton invented the term "associative" at that moment. Therefore, one can say that the octonions were essential to enlighten the notion of associativity in Algebra. Then, Hamilton offered himself to publicize Graves' discovery. However, since he was always engaged with the quaternions he had just created, Hamilton kept postponing such offering.

In the meantime, the young *Arthur Cayley* (1821 - 1895) was thinking on the quaternions since Hamilton announced their existence. On March, 1845, he published an article on the *Philosophical Magazine* entitled "On Jacobi's Elliptic Functions, in Reply to the Rev. B. Bronwin; and on Quaternions". In a significant part of this article, Cayley tried to refute another paper, in which the author pointed out errors in his work on elliptic functions. Apparently, Cayley gave a brief description of the octonions in this work. In fact, Cayley's article was so full of errors that it was omitted from his collected works, with the exception of the part in which he treated the octonions.

Annoyed with being beaten to publication, Graves attached a postscript in one of his articles who would appear on the next edition of the *Philosophical Magazine* saying that he knew about the octonions since the Christmas of 1843. On June 14th, 1847, Hamilton wrote a small note to the *Transactions of The Royal Irish Academy* alleging Graves' pioneerism. Nonetheless, it was too late, the octonions had already entered history as *Cayley's numbers*. The interested reader can find more details in [2].

Remark 4.5. The following facts hold true.

- $\mathbb{R}$ is an associative and commutative real division algebra. Evidently, the proofs of these assertions strongly depend on the formalization that one chooses for the real numbers.
- $\mathbb{C}$ is an associative and commutative real division algebra.
- $\mathbb{H}$ is an associative and non-commutative real division algebra. The relations $e_1e_2 = e_3$ and $e_2e_1 = -e_3$ prove its non-commutativity, while the associativity follows from straightforward computations.
- $\mathbb{O}$ is an alternative, non-associative and non-commutative real division algebra. In fact, it is non-commutative because $e_1e_2 = e_4$ and $e_2e_1 = -e_4$. Moreover, it is non-associative because $(e_1e_2)e_3 = e_4e_3 = -e_6$ and $e_1(e_2e_3) = e_1e_5 = e_6$. At this level, the proof of the alternance of the octonions are just cumbersome computations. $\diamondsuit$

All of the division algebras of this section have multiplicative inverses. Indeed, with the exception of the real numbers in which we have to prove the existence of inverses by means of a formalization, all of the proofs are straightforward computations. Furthermore, we have that all of these algebras are normed with respect to the canonical Euclidean norm.

In the next section, we follow Graves' path to construct as many algebras as one would like to have. The process to be described is the *Cayley-Dickson construction*, named after Arthur Cayley, whose name baptizes the octonions, and *Leonard Dickson* (1874-1954), that showed in 1919 how the octonions could be obtained as a two-dimensional algebra over the quaternions. In order to be historically fair, maybe it would be better to say "Graves-Dickson construction" or even "Graves-Cayley-Dickson construction".

5. Star-algebras and the Cayley-Dickson construction

In this section, we introduce star-algebras and some terminology that comes together. This is done in order to talk about the Cayley-Dickson construction, which unifies the algebras discussed above and introduces questions that send us to the classical theorems of the following section.

Definition 5.1. A **real star-algebra** is a pair $(\mathcal{A}, *)$ in which:

- $\mathcal{A}$ is a real algebra;
- $* : \mathcal{A} \to \mathcal{A}$ is an **anti-involution** of $\mathcal{A}$, that is, an anti-isomorphism whose inverse coincides with itself. ♢

Example 5.2. We have the following examples of star-algebras.

- $\mathbb{R}$ is a real division star-algebra with respect to the anti-involution $* : \mathbb{R} \to \mathbb{R}$ given by $\alpha^* = \alpha$.
- $\mathbb{C}$ is a real division star-algebra with respect to the anti-involution $* : \mathbb{C} \to \mathbb{C}$ given by $(\alpha + \alpha_1 e_1)^* = \alpha - \alpha_1 e_1$.
- $\mathbb{H}$ is a real division star-algebra with respect to the anti-involution $* : \mathbb{H} \to \mathbb{H}$ given by $(\alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3)^* = \alpha - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3$.
- $\mathbb{O}$ is a real division star-algebra with respect to the anti-involution $* : \mathbb{O} \to \mathbb{O}$ given by $(\alpha + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7)^* = \alpha - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3 - \alpha_4 e_4 - \alpha_5 e_5 - \alpha_6 e_6 - \alpha_7 e_7$. ♢

Definition 5.3. A star-algebra $\mathcal{A}$ is **nicely normed** provided that:

- the sum $a + a^*$ is a real multiple of $1 \in \mathcal{A}$ for all $a \in \mathcal{A}$;
- the products aa^* and a^*a are equal, a positive multiple of $1 \in \mathcal{A}$ for all non-zero $a \in \mathcal{A}$. ♢

Remark 5.4. The following facts hold true.

- If $\mathcal{A}$ is a nicely normed real star-algebra, then it has multiplicative inverses. Indeed, it suffices to see that, for every non-zero element $a \in \mathcal{A}$, the inverse a^{-1} of a is given by
$$a^{-1} = \frac{1}{aa^*} a^*.$$
- If $\mathcal{A}$ is nicely normed and alternative, then it is a normed real algebra. In fact, we define the norm
$$\begin{aligned} |\cdot| : \mathcal{A} &\to [0, \infty) \\ a &\mapsto \sqrt{aa^*}. \end{aligned}$$
We claim that $|a|\,|b| = |ab|$ for all $a, b \in \mathcal{A}$. Indeed, since $\mathcal{A}$ is alternative, we have $|ab|^2 = (ab)(ab)^* = ab(b^*a^*) = a(bb^*)a^* = aa^*\,|b|^2 = |a|^2\,|b|^2$ for all $a, b \in \mathcal{A}$. ♢

Definition 5.5. Let $(\mathcal{A}, *)$ be a real star-algebra. The **Cayley-Dickson algebra** of $\mathcal{A}$, denoted by $\mathrm{CD}(\mathcal{A})$:

- as a vector space, is the direct sum $\mathcal{A} \oplus \mathcal{A}$;
- as a real algebra, has the multiplication $\mathrm{CD}(\mathcal{A}) \times \mathrm{CD}(\mathcal{A}) \to \mathrm{CD}(\mathcal{A})$ given by $(a, b)(c, d) = (ac - db^*, a^*d + cb)$;
- as a real star-algebra, has the anti-involution $* : \mathrm{CD}(\mathcal{A}) \to \mathrm{CD}(\mathcal{A})$ given by $(a, b)^* = (a^*, -b)$. ♢

Proposition 5.6. *Let $\mathcal{A}$ be a real star-algebra. The following facts hold true.*

1. *$\mathcal{A}$ is nicely normed if and only if $\mathrm{CD}(\mathcal{A})$ is nicely normed.*
2. *$\mathcal{A}$ is associative and nicely normed if and only if $\mathrm{CD}(\mathcal{A})$ is alternative and nicely normed.*

Proof. We compute to prove these facts as below.

1. If $\mathrm{CD}(\mathcal{A})$ is nicely normed, then $\mathcal{A}$ is also nicely normed since it is canonically a subalgebra of $\mathrm{CD}(\mathcal{A})$. On the other hand, we assume that $\mathcal{A}$ is nicely normed. If $(a, b) \in \mathrm{CD}(\mathcal{A})$, then

$$(a, b) + (a, b)^* \;=\; (a + a^*, 0)$$

is a real multiple of $(1, 0) \in \mathrm{CD}(\mathcal{A})$ since $a + a^*$ is a real multiple of $1 \in \mathcal{A}$. Moreover,

$$(a, b)(a, b)^* \;=\; (aa^* + bb^*, 0) \;=\; (a^*a + b^*b, 0) \;=\; (a, b)^*(a, b)$$

because $aa^* = a^*a$ and $bb^* = b^*b$ in $\mathcal{A}$. Finally, if (a, b) is non-zero in $\mathrm{CD}(\mathcal{A})$, then

$$(a, b)(a, b)^* \;=\; (aa^* + bb^*, 0)$$

is a positive multiple of $(1, 0) \in \mathrm{CD}(\mathcal{A})$ since aa^* and bb^* are non-negative multiples of $1 \in \mathcal{A}$ with at least one of them positive. These facts ensure that $\mathrm{CD}(\mathcal{A})$ is nicely normed.

2. Suppose A is associative and nicely normed. We know that $\mathrm{CD}(\mathcal{A})$ is nicely normed, thence we just have to prove that $\mathrm{CD}(\mathcal{A})$ is alternative. In other words, we have to verify the two conditions presented in Definition 3.2. We leave the second condition to the reader, but we compute to prove the first one. Indeed, we have

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$$\begin{aligned}
(a,b)^2(c,d) &= (aa-bb^*, a^*b+ab)(c,d)\\
&= ((aa-bb^*)c-d(a^*b+ab)^*, (aa-bb^*)^*d+c(a^*b+ab))\\
&= (aac-bb^*c-db^*a-db^*a^*, a^*a^*d-b^*bd+ca^*b+cab)\\
&= (aac-bb^*c-db^*(a+a^*), a^*a^*d-b^*bd+c(a^*+a)b)\\
&= (aac-(a+a^*)db^*-cbb^*, a^*a^*d+(a^*+a)cb-db^*b)\\
&= (aac-adb^*-a^*db^*-cbb^*, a^*a^*d+a^*cb+acb-db^*b)\\
&= (a(ac-db^*)-(a^*d+cb)b^*, a^*(a^*d-cb)+(ac-db^*)b)\\
&= (a,b)(ac-db^*, a^*d+cb)\\
&= (a,b)[(a,b)(c,d)].
\end{aligned}$$

Hence, $\mathrm{CD}(\mathcal{A})$ is alternative. The converse is somehow analogous since it suffices to reverse the logic applied to the computation above. For conciseness, we leave the details to the reader. □

Remark 5.7. One can prove

$$\mathrm{CD}(\mathbb{R}) = \mathbb{C}, \qquad \mathrm{CD}(\mathbb{C}) = \mathbb{H} \qquad \text{and} \qquad \mathrm{CD}(\mathbb{H}) = \mathbb{O}.$$

Thence, through the Cayley-Dickson construction, we can see that the historical algebras descend from the real numbers. Moreover, we can continue applying such procedure to obtain an infinite family of real algebras, each of which with dimension equal to a power of two. For instance, we obtain the **real algebra of the sedenions** as the Cayley-Dickson algebra

$$\mathbb{S} = \mathrm{CD}(\mathbb{O}).$$

The sedenions are our first example of a real algebra with zero divisors. Because of that, neither they can be normed nor they form an alternative algebra. In fact, since $\mathbb{O}$ and $\mathbb{S}$ are nicely normed, Proposition 5.6 says that $\mathbb{S}$ is alternative if and only if $\mathbb{O}$ is associative, but $\mathbb{O}$ is not associative. ♢

In general, if and algebra $\mathcal{A}$ has zero divisors, then $\mathrm{CD}(\mathcal{A})$ has also zero divisors. Furthermore, if $\mathcal{A}$ is not alternative, then $\mathrm{CD}(\mathcal{A})$ cannot be alternative. Thus, the inductive application of the Cayley-Dickson construction from the sedenions results in non-alternative algebras with zero divisors. In particular, these algebras cannot be normed because of Remark 3.3. In the next section, we explore this myriad of algebras given by the Cayley-Dickson construction to ask questions about existence of zero divisors, associativity and normability in algebras. As answers to these questions, many classical results appear, some of them with proofs. However, the main result will have to wait until the last section.

6. Classical theorems

In the preceding section, the Cayley-Dickson construction has left a natural question to be answered now: Is it possible to change the multiplication of the sedenions and, more generally, of the Cayley-Dickson algebras that come after, in order to turn them into division algebras? The answer for this question is contained in the following result, which is due to *Raoul Bott* (1923-2005), *John Milnor* (1931-) and *Michel Kervaire* (1927-2007).

Theorem 6.1 (Bott-Milnor-Kervaire Theorem)**.** *Every real division algebra has dimension* 1, 2, 4 *or* 8.

This result was independently proved by Bott-Milnor and by Kervaire in 1958, according to [2, p. 150]. We will give a sketch of its proof in the last section, following [5, pp. 59-72]. Complementarily, we have proved that there exist real division algebras in dimensions 1, 2, 4 and 8. Indeed, we have $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ in Examples 4.1, 4.2, 4.3 and 4.4, respectively. Nevertheless, these algebras are not the only real division algebras in these dimensions up to isomorphism (with the natural exception of the real numbers). In fact:

- in dimension 2, we have the hyperbolic complex numbers by declaring $e_1^2 = 1$.
- in dimension 4, we have the quaternionic algebra defined by declaring $e_1^2 = e_2 - 1$;
- in dimension 8, we have the Cayley-Dickson algebra of the preceding quaternionic algebra.

Therefore, we could ask if the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are also special from a strictly mathematical viewpoint. Subsequently, we present positive answers for this question. We begin with the following technical lemma, that helps proving the *Frobenius Theorem*.

Lemma 6.2. *Let $\mathcal{A}$ be an n-dimensional real division algebra. Then*

$$\Xi \ = \ \{a \in \mathcal{A} \,:\, a^2 \leqslant 0\}$$

is an $(n-1)$-dimensional vector subspace of $\mathcal{A}$ such that $\mathcal{A} = \mathbb{R} \oplus \Xi$. This implies that $\mathcal{A}$ is generated by Ξ as an algebra.

Proof. Any element of $\mathcal{A}$ defines an endomorphism of $\mathcal{A}$ by left-multiplication, so we can identify it with that endomorphism and speak of its trace, characteristic and minimal polynomials. Let $a \in \mathcal{A}$ and let $p(x)$ be its characteristic polynomial. We have from the *Fundamental Theorem of Algebra* that there exists $x_1, \ldots, x_r \in \mathbb{R}$ and $z_1, \ldots, z_s \in \mathbb{C} - \mathbb{R}$ such that

$$p(x) = (x - x_1) \cdots (x - x_r) q(x, z_1) \cdots q(x, z_s) \in \mathbb{R}[x]$$

where $q(x, z) = (x - z)(x - \bar{z}) \in \mathbb{R}[x]$ for any $z \in \mathbb{C}$. We have $p(a) = 0$ by the *Cayley-Hamilton Theorem*. Thence, since $\mathcal{A}$ is a division algebra, either $a - x_i = 0$ for some i between 1 and r, both included, or $q(x, z_j) = 0$ for some j between 1 and s, both included. The first possibility implies that a is a real multiple of $1 \in \mathcal{A}$. In turn, since $q(x, z_j)$ is irreducible over the reals, the second one implies that $q(x, z_j)$ is the minimal polynomial of $a \in \mathcal{A}$. Because $p(x)$ is real and has the same complex roots as the minimal polynomial,

$$p(x) = q(x, z_j)^{\ell}$$

for a certain $\ell \in \mathbb{N}$. It is well-known that the coefficient of $x^{2\ell-1}$ in $p(x)$ is tr(a) up to sign. Therefore, if we write

$$q(x, z_j) = x^2 - 2\Re(z_j)x + |z_j|^2,$$

then it is clear that tr$(a) = 0$ if and only if $\Re(z_j) = 0$. Equivalently, tr$(a) = 0$ if and only if $a^2 = -|z_j|^2 < 0$. Therefore,

$$\Xi = \{a \in \mathcal{A} : \text{tr}(a) = 0\}.$$

In particular, Ξ is a vector subspace of $\mathcal{A}$. Moreover, the *Rank-Nullity Theorem* says that Ξ has dimension $n - 1$ since it is the kernel of tr : $\mathcal{A} \to \mathbb{R}$. Finally, it is obvious that $\mathcal{A} = \mathbb{R} \oplus \Xi$. □

Theorem 6.3 (Frobenius Theorem)**.** *The only associative real division algebras are* $\mathbb{R}$*,* $\mathbb{C}$ *and* $\mathbb{H}$*.*

Proof. We use the same notation of the preceding lemma, but now we suppose that $\mathcal{A}$ is associative. We define

$$\begin{aligned} Q : \Xi \times \Xi &\to \Xi \\ (a, b) &\mapsto -\frac{1}{2}(ab + ba). \end{aligned}$$

It is clear that Q is real since

$$Q(a, b) = \frac{a^2 + b^2 - (a + b)^2}{2}$$

for all $a, b \in \Xi$. Furthermore, it is obvious that $Q(a, a) > 0$ for all $a \in \Xi - \{0\}$. As a consequence, Q induces a positive definite symmetric bilinear form on Ξ, that is, an inner product on Ξ.

Let Θ be a minimal subspace of Ξ that generates $\mathcal{A}$ as an algebra and let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of Θ with respect to Q. From orthonormality, we have $e_i^2 = -1$ for all i between 1 and m, both included, and $e_i e_j = -e_j e_i$ for all distinct i and j between 1 and m, both included. Therefore, we analyze the following situations:

- if $m = 0$, then $\mathcal{A}$ is generated by 1, and thus isomorphic to $\mathbb{R}$;
- if $m = 1$, then $\mathcal{A}$ is generated by 1 and e_1 subject to $e_1^2 = -1$, and thus isomorphic to $\mathbb{C}$;
- if $m = 2$, then $\mathcal{A}$ is generated by 1, e_1 and e_2 subject to $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$, and thus isomorphic to $\mathbb{H}$ since the third imaginary part of the quaternions coincides with the product of the other two;
- if $m > 2$, then $v = e_1 e_2 e_m$ is well-defined ($\mathcal{A}$ is associative). We have

$$v^2 = e_1 e_2 e_m e_1 e_2 e_m = -e_1^2 e_2^2 e_m^2 = 1.$$

Thus,

$$(v+1)(v-1) = v^2 - 1 = 0.$$

Since $\mathcal{A}$ is a division algebra, this implies $v = 1$ or $v = -1$. In turn,

$$e_1 e_2 = -e_m \qquad \text{or} \qquad e_1 e_e = e_m$$

respectively. Therefore, contradicting the minimality of Θ, we have that $\{e_1, \ldots, e_{m-1}\}$ generate $\mathcal{A}$. Consequently, as we wished, this situation is not possible. □

The reader can find another interesting proof of the Frobenius Theorem in [8], that the author claims to be a "self-contained proof which seems both elementary and conceptual". There is also a fantastic result called *Zorn's Theorem*: the only alternative real division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. This was proved by *Max Zorn* (1906-1993), the guy of the lemma, in the paper [11] of 1930, that was correlated to his doctoral thesis. We will not prove it here, but we point out that the Frobenius Theorem follows easily from Zorn's Theorem.

Finally, we sketch the proof of the fact that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are also the only normed division algebras. This result was first proved by *Adolf Hurwitz* (1859-1919) in the paper [6] of 1898. In the sketch given below, we assume some knowledge on Clifford algebras and their representations. The reader can find all of the notions required for this in [4, pp. 149-181].

Theorem 6.4 (Hurwitz's Theorem)**.** *The only normed real division algebras are* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$ *and* $\mathbb{O}$*.*

Sketch of proof. It can be proved that every n-dimensional normed division algebra has an n-dimensional representation of the Clifford algebra Cliff$(n-1)$. The periodicity of Clifford algebras implies that there exists such a representation of Cliff$(n-1)$ only if $n = 1, 2, 4$ or 8. This happens because the irreducible representations of Cliff$(n+8)$ are obtained by tensoring those of Cliff(n) with $\mathbb{R}^{16}$. But since this multiplies dimension by 16, the irreducible representations of Cliff$(n-1)$ always have dimension greater than n if $n > 8$. Then, by checking the representations of Cliff(n) for $n \leqslant 8$, we exclude the undesired dimensions.

We have $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, so normed division algebras exist in the dimensions mentioned above. The only remaining question is about the uniqueness of these algebras. This demands one to investigate more deeply the relation between normed division algebras and the Cayley-Dickson construction. The interested reader can find some details in [2, p. 158]. □

In the next section, we use topological K-theory to prove the Bott-Milnor-Kervaire Theorem. Therefore, since we used Clifford algebras above, it is interesting to note that there is a connection between Clifford algebras and K-Theory, the *Atiyah-Bott-Shapiro Theorem*. This result also relates the periodicity of Clifford algebras to the one of K-Theory given by the *Bott Periodicity Theorem*. These periodicity properties are in the heart of many applications, including the Hurwitz's Theorem and the Bott-Milnor-Kervaire Theorem.

7. The Bott-Milnor-Kervaire Theorem

In this last section, we exhibit the main ideas involved in the proof of Theorem 6.1, whose historical and mathematical relevance we hope to have shown to the reader in the preceding discussions. In order to keep the reader curious, maybe it it worth mentioning that, in the end, we will need the following elementary lemma from *Number Theory*.

Lemma 7.1. *Let n be a natural number. If 2^n divides $3^n - 1$, then n must be either* 1*,* 2 *or* 4*.*

Proof. We first write $n = 2^\ell m$ with m an odd number, because of the *Fundamental Theorem of Arithmetic*. Then, it suffices to show that the highest power of 2 that divides $3^n - 1$ is 2^1 for $\ell = 0$ and $2^{\ell+2}$ for $\ell > 0$. Indeed, if 2^n divides $3^n - 1$, then $n \leqslant \ell + 2$ by this fact. Hence,

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$$2^\ell \leqslant 2^\ell m = n \leqslant \ell + 2$$

implies $\ell \leqslant 2$ and $n \leqslant \ell + 2 \leqslant 4$. Then, by checking the cases, we exclude $n = 3$ as desired. Therefore, let us find the highest power of 2 dividing $3^n - 1$. We do it by induction on ℓ.

- For $\ell = 0$, we just have to prove that $3^n - 1$ is not divisible by $2^2 = 4$. In fact, since m is odd, $3^n - 1 = 3^m - 1 \equiv (-1)^m - 1 \equiv 2 \pmod 4$.
- For $\ell = 1$, we just have to prove that $3^m + 1$ is not divisible by $2^3 = 8$ because $3^n - 1 = 3^{2m} - 1 = (3^m - 1)(3^m + 1)$ and the highest power of 2 dividing $3^m - 1$ is 2^1 as above. In fact, $3^m + 1 = 3^{2\left(\frac{m-1}{2}\right)+1} + 1 \equiv 1^{\frac{m-1}{2}} \cdot 3 + 1 = 4 \pmod 8$ since m is odd.
- The inductive step is the same as passing from n to $2n$ with n even. In this case, we write $3^{2n} - 1 = (3^n - 1)(3^n + 1)$. We have $3^n + 1 \equiv (-1)^n + 1 = 2 \pmod 4$ since n is even, so the highest power of 2 that divides $3^n + 1$ is 2^1. Thus, the highest power of 2 dividing $3^{2n} - 1$ is twice the highest power of 2 dividing $3^n - 1$, as desired. □

Now we begin the actual path to the proof by first reminding the reader that the ***n*-sphere** is

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^{n+1}$. It is equipped with the subspace topology of $\mathbb{R}^{n+1}$. Moreover, we say that the n-sphere $\mathbb{S}^n$ is an **H-space** if there exists a continuous binary operation $\mu : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$ with a two-sided identity[3]. These notions are important because the contrapositive of the following lemma is a key step for proving the desired result.

Lemma 7.2. *Let n be a natural number. If $\mathbb{R}^{n+1}$ is a real division algebra, then $\mathbb{S}^n$ is an H-space.*

Proof. If $\mathbb{R}^{n+1}$ is a division algebra, then $\mu : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$ is a continuous binary operation with a two-sided identity for $\mu(x, y) = \frac{xy}{|xy|}$. □

From the preceding lemma, in order to prove the desired result, it suffices to show that $\mathbb{S}^n$ is an H-space only if $n = 0$, 1, 3 or 7. This happens because we can prove the theorem only when the underlying vector space is $\mathbb{R}^{n+1}$. Indeed, if $\mathcal{A}$ is an

[3]In general, a topological space being an H-space is weaker than it being a topological group. This happens because the first notion does not require associativity and inverses for the binary operation, while the second one does require these properties. For example, $\mathbb{S}^1$ and $\mathbb{S}^3$ are topological groups with the multiplications induced from $\mathbb{C}$ and $\mathbb{H}$, respectively. In turn, $\mathbb{S}^7$ is an H-space with the multiplication induced from $\mathbb{O}$. However, it is not a topological group since the multiplication lacks associativity, as seen in Remark 4.5.

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$(n+1)$-dimensional vector space, then let $\alpha : \mathcal{A} \to \mathbb{R}^{n+1}$ be a linear isomorphism. The diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{m_{\mathcal{A}}} & \mathcal{A} \\ \Big\downarrow{\scriptstyle \alpha\times\alpha} & & \Big\downarrow{\scriptstyle \alpha} \\ \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} & \xrightarrow[m_{n+1}]{} & \mathbb{R}^{n+1} \end{array}$$

proves our assertion since:

- if $m_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a division algebra structure on $\mathcal{A}$, then $\alpha \circ m_{\mathcal{A}} \circ (\alpha \times \alpha)^{-1}$ is a division algebra structure on $\mathbb{R}^{n+1}$;
- if $m_{n+1} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a division algebra structure on $\mathbb{R}^{n+1}$, then $\alpha^{-1} \circ m_{n+1} \circ (\alpha \times \alpha)$ is a division algebra structure on $\mathcal{A}$.

To prove that the sufficient condition presented above holds true, we need K-Theory techniques that we do not present here for conciseness. The interested reader can find all of the necessary information in [1], [4] and [5]. To be as direct as possible, let us show that $\mathbb{S}^{2k}$ *is not an H-space for all* $k \in \mathbb{N}$. This is the first part of the proof of Theorem 6.1.

Proof. The absolute K-Theory group $K(\mathbb{S}^{2n})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ for every $n \in \mathbb{N}$. The external product

$$\boxtimes : \; K(\mathbb{S}^{2k}) \otimes K(X) \; \to \; K(\mathbb{S}^{2k} \times X)$$

is an isomorphism. Then, it follows from the fact that $K(\mathbb{S}^{2k})$ can be described as the quotient ring $\mathbb{Z}[\gamma]/(\gamma^2)$ that

$$K(\mathbb{S}^{2k} \times \mathbb{S}^{2l}) \; \simeq \; \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$$

where α and β are the pullbacks of the generators of the reduced K-Theory groups $\widetilde{K}(\mathbb{S}^{2k})$ and $\widetilde{K}(\mathbb{S}^{2l})$ under the natural projections of the Cartesian product $\mathbb{S}^{2k} \times \mathbb{S}^{2l}$. As a consequence, $\{1, \alpha, \beta, \alpha\beta\}$ is an additive basis for $K(\mathbb{S}^{2k} \times \mathbb{S}^{2l})$. Now we assume that there exists an H-space multiplication $\mu : \mathbb{S}^{2k} \times \mathbb{S}^{2k} \to \mathbb{S}^{2k}$. We claim that the induced homomorphism of K-rings

$$K(\mu) : \; \mathbb{Z}[\gamma]/(\gamma^2) \; \to \; \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$$

is such that

$$K(\mu)(\gamma) \; = \; \alpha + \beta + m\alpha\beta$$

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for some $m \in \mathbb{Z}$. Indeed,

$$\mathbb{S}^{2k} \xrightarrow{\quad i \quad} \mathbb{S}^{2k} \times \mathbb{S}^{2k} \xrightarrow{\quad \mu \quad} \mathbb{S}^{2k}$$

is the identity, where i is the inclusion into $\mathbb{S}^{2k} \times \{e\}$ with e being the identity element of the H-space structure μ. Thus, $K(i)$ for i the inclusion onto the first factor sends α to γ and β to zero. Consequently, the coefficient of α in $K(\mu)(\gamma)$ must be 1. In a similar manner, the coefficient of β in $K(\mu)(\gamma)$ must also be 1. However, this is impossible since it would imply

$$K(\mu)(\gamma^2) \;=\; (\alpha + \beta + m\alpha\beta)^2 \;=\; 2\alpha\beta$$

despite

$$K(\mu)(\gamma^2) \;=\; 0$$

since $\gamma^2 = 0$. □

We still have to prove that, *if k is a natural different from 1, 2 or 4, then $\mathbb{S}^{2k-1}$ is not an H-space*. This is the hard part of the proof. The idea is to associate a map $\mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ to any map $\mathbb{S}^{2k-1} \times \mathbb{S}^{2k-1} \to \mathbb{S}^{2k-1}$, and then show that the *Hopf invariant* of the first one is equal to plus or minus the unit provided that the second one is an H-space multiplication. Consequently, the problem is solved by proving that a map $\mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ has Hopf invariant equal to plus or minus the unit only if $k = 1, 2$ or 4. For this, one has to show the existence of a special kind of ring homomorphism in K-Theory, *Adams operations*.

We begin by defining $\widehat{\varphi} : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ from $\varphi : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. For this, we regard $\mathbb{S}^{2n-1}$ as

$$\partial(\mathbb{D}^n \times \mathbb{D}^n) \;=\; \partial\mathbb{D}^n \times \mathbb{D}^n \cup \mathbb{D}^n \times \partial\mathbb{D}^n$$

where

$$\mathbb{D}^n \;=\; \{x \in \mathbb{R}^n \,:\, |x| \leqslant 1\}$$

is the ***n*-disc** equipped with the subspace topology of $\mathbb{R}^n$. We also regard $\mathbb{S}^n$ as the union of two disks $\mathbb{D}^n_+$ and $\mathbb{D}^n_-$ with their boundaries identified. Figures 3 and 4 may help the visualization. We define

$$\widehat{\varphi}(x,y) \;=\; \begin{cases} |y| \cdot \varphi(x, \frac{y}{|y|}) \in \mathbb{D}^n_+ \text{ for } (x,y) \in \partial\mathbb{D}^n \times \mathbb{D}^n \\ |x| \cdot \varphi(\frac{x}{|x|}, y) \in \mathbb{D}^n_- \text{ for } (x,y) \in \mathbb{D}^n \times \partial\mathbb{D}^n. \end{cases}$$

It is not difficult to check that $\widehat{\varphi}$ is well-defined and continuous. Moreover, we have $\widehat{\varphi} = \varphi$ on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

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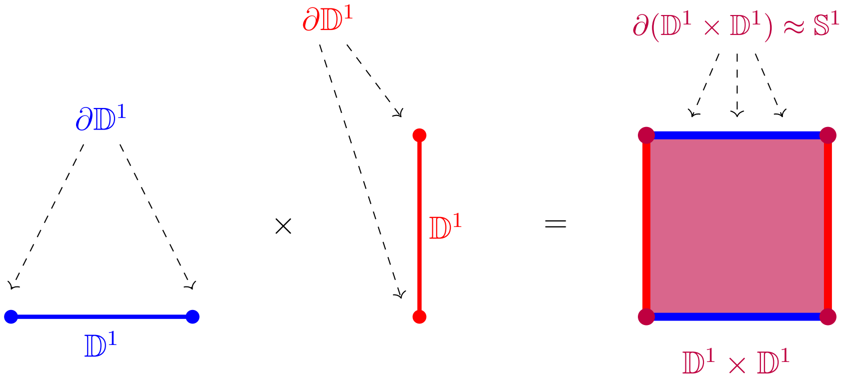

Figure 3. Visual representation of the circle $\mathbb{S}^1$ as the boundary of the product $\mathbb{D}^1 \times \mathbb{D}^1$. The decomposition of this boundary as $\partial\mathbb{D}^1 \times \mathbb{D}^1 \cup \mathbb{D}^1 \times \partial\mathbb{D}^1$ is clear since $\partial\mathbb{D}^1 \times \mathbb{D}^1$ coincides with the red vertical segments of the boundary of the square while $\mathbb{D}^1 \times \partial\mathbb{D}^1$ coincides with the blue horizontal segments of the boundary of the square.

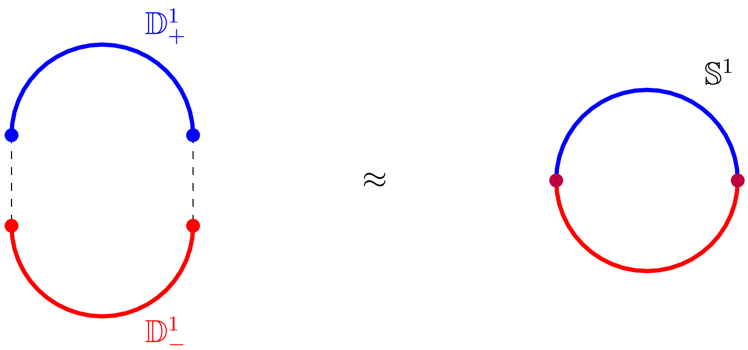

Figure 4. Visual representation of the circle $\mathbb{S}^1$ as the union of the blue disc $\mathbb{D}^1_+$ and the red disc $\mathbb{D}^1_-$ with the boundaries identified. The purple points on the right give the idea of identifying the blue and red points on the left, that are the boundaries of the discs.

Now we consider the case in which n is even, so we replace n by $2k$. Let C_f be the space $\mathbb{S}^{2k}$ with a $4k$-dimensional cell e^{4k} attached by a map $f : \mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$. More explicitly,

$$C_f \;=\; \mathbb{S}^{2k} \sqcup e^{4k} / \sim$$

with $x \in \partial e^{4k} = \mathbb{S}^{4k-1}$ identified with $f(x) \in \mathbb{S}^{2k}$. The quotient $C_f/\mathbb{S}^{2k}$ is then $\mathbb{S}^{4k}$, and since $\widetilde{K}^1(\mathbb{S}^{4k}) = \widetilde{K}^1(\mathbb{S}^{2k}) = 0$, the exact sequence of the pair $(C_f, \mathbb{S}^{2k})$ becomes a short exact sequence

$$0 \longrightarrow \widetilde{K}(\mathbb{S}^{4k}) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(\mathbb{S}^{2k}) \longrightarrow 0.$$

Denoting by H the Boot bundle that gives the isomorphism in the *Bott Periodicity Theorem*, let $\alpha \in \widetilde{K}(C_f)$ be the image of the generator $(H-1)^{4k}$ of $\widetilde{K}(\mathbb{S}^{4k})$ and let

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$\beta \in \widetilde{K}(C_f)$ map to the generator $(H-1)^{2k}$ of $\widetilde{K}(\mathbb{S}^{2k})$. We have that the element β^2 maps to zero in $\widetilde{K}(\mathbb{S}^{2k})$ since the square of any element of $\widetilde{K}(\mathbb{S}^{2k})$ is zero. Thence, by exactness

$$\beta^2 \; = \; h\alpha$$

for some integer h, called the **Hopf invariant**[4] of f. We have all of the machinery needed for the following lemma.

Lemma 7.3. *If $\mu : \mathbb{S}^{2k-1} \times \mathbb{S}^{2k-1} \to \mathbb{S}^{2k-1}$ is an H-space multiplication, then the associated map $\widehat{\mu} : \mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ necessarily has Hopf invariant equal to plus or minus the unit.*

Proof. Let $e \in \mathbb{S}^{2k-1}$ be the identity element for the H-space multiplication μ. In view of the definition of $\widetilde{\mu}$ it is natural to view the characteristic map Φ of the $4k$-cell of $C_{\widehat{\mu}}$ as a map

$$(\mathbb{D}^{2k} \times \mathbb{D}^{2k}, \partial(\mathbb{D}^{2k} \times \mathbb{D}^{2k})) \to (C_{\widehat{\mu}}, \mathbb{S}^{2k}).$$

In the following commutative diagram, the horizontal maps are the product maps. In turn, the diagonal map is external product, that is equivalent to the external product $\widetilde{K}(\mathbb{S}^{2k}) \otimes \widetilde{K}(\mathbb{S}^{2k}) \to \widetilde{K}(\mathbb{S}^{4k})$, which is an isomorphism since it is an iterate of the Bott periodicity isomorphism.

$$\begin{array}{ccc}
\widetilde{K}(C_{\widehat{\mu}}) \otimes \widetilde{K}(C_{\widehat{\mu}}) & \longrightarrow & \widetilde{K}(C_{\widehat{\mu}}) \\
\approx \uparrow & & \uparrow \\
\widetilde{K}(C_{\widehat{\mu}}, \mathbb{D}^{2k}_-) \otimes \widetilde{K}(C_{\widehat{\mu}}, \mathbb{D}^{2k}_+) & \longrightarrow & \widetilde{K}(C_{\widehat{\mu}}, \mathbb{S}^{2k}) \\
{\scriptstyle \widetilde{K}(\Phi)\otimes\widetilde{K}(\Phi)} \downarrow & & \downarrow {\scriptstyle \widetilde{K}(\Phi)} \\
\widetilde{K}(\mathbb{D}^{2k} \times \mathbb{D}^{2k}, \partial\mathbb{D}^{2k} \times \mathbb{D}^{2k}) \otimes \widetilde{K}(\mathbb{D}^{2k} \times \mathbb{D}^{2k}, \mathbb{D}^{2k} \times \partial\mathbb{D}^{2k}) & \longrightarrow & \widetilde{K}(\mathbb{D}^{2k} \times \mathbb{D}^{2k}, \partial(\mathbb{D}^{2k} \times \mathbb{D}^{2k})) \\
\approx \downarrow & \nearrow \approx & \\
\widetilde{K}(\mathbb{D}^{2k} \times e, \partial\mathbb{D}^{2k} \times e) \otimes \widetilde{K}(e \times \mathbb{D}^{2k}, e \times \partial\mathbb{D}^{2k}) & &
\end{array}$$

[4]To see that h is well-defined, that is, independent of the choice of β, note that β is unique up to adding a multiple of α, and $(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta$ since $\alpha^2 = 0$, so it suffices to show that $\alpha\beta = 0$. Indeed, since α maps to zero in $\widetilde{K}(\mathbb{S}^{2k})$, so does $\alpha\beta$, hence $\alpha\beta = \ell\alpha$ for some integer ℓ. Multiplying the equation $\ell\alpha = \alpha\beta$ on the right by β gives $\ell\alpha\beta = \alpha\beta^2 = \alpha(h\alpha) = h\alpha^2$, and this is zero since $\alpha^2 = 0$. Thus $\ell\alpha\beta = 0$, which implies $\alpha\beta = 0$ since $\alpha\beta$ lies in an infinite cyclic subgroup of $\widetilde{K}(C_f)$, the image of $\widetilde{K}(\mathbb{S}^{4k})$.

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By the definition of an H-space and the definition of $\widehat{\mu}$, the map Φ restricts to a homeomorphism from $\mathbb{D}^{2k} \times e$ onto $\mathbb{D}^{2k}_{-}$ and from $e \times \mathbb{D}^{2k}$ onto $\mathbb{D}^{2k}_{+}$. Therefore, $\beta \otimes \beta \in \widetilde{K}(C_{\widehat{\mu}}) \otimes \widetilde{K}(C_{\widehat{\mu}})$[5] maps to a generator in the bottom row of the diagram, since β restricts to a generator of $\widetilde{K}(\mathbb{S}^{2k})$. By commutativity of the diagram, the product map in the top row sends $\beta \otimes \beta$ to $\pm\alpha$ since α is the image of a generator of $\widetilde{K}(C_{\widehat{\mu}}, \mathbb{S}^{2k})$. Consequently, we have $\beta^2 = \pm\alpha$. This precisely says that the Hopf invariant of $\widehat{\mu}$ is plus or minus the unit. □

According to the outline given above and to all of the details we have already proven, in order to conclude the proof of the Bott-Milnor-Kervaire Theorem, it suffices to show that *there exists a map* $\mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ *with Hopf invariant equal to plus or minus the unit only if* $k = 1, 2$ *or* 4. This result, due to *John Frank Adams* (1930-1989), is proved below to close section.

Proof. John Adams proved that, for every compact Hausdorff space X, there exist ring homomorphisms

$$\psi^{\ell}_{X} : \ K(X) \ \to \ K(X)$$

such that the diagram

$$\begin{array}{ccccc}
 & & \overset{\psi^{\ell m}_X}{\curvearrowright} & & \\
K(X) & \xrightarrow{\psi^{\ell}_X} & K(X) & \xrightarrow{\psi^{m}_X} & K(X) \\
\uparrow{\scriptstyle K(g)} & & \uparrow{\scriptstyle K(g)} & & \uparrow{\scriptstyle K(g)} \\
K(Y) & \xrightarrow[\psi^{\ell}_Y]{} & K(Y) & \xrightarrow[\psi^{m}_Y]{} & K(Y) \\
 & & \underset{\psi^{\ell m}_Y}{\smile} & &
\end{array}$$

is commutative for any map $g : X \to Y$ between compact Hausdorff spaces,

$$\psi^{\ell}(\theta) \ = \ \theta^{\ell}$$

if θ is a line bundle, and

$$\psi^{p}(\eta) \ \equiv \ \eta^{p} \quad (\text{mod } p)$$

[5]Here $\alpha \in \widetilde{K}(C_{\widehat{\mu}})$ is the image of the generator $(H-1)^{4k}$ of $\widetilde{K}(\mathbb{S}^{4k})$ and $\beta \in \widetilde{K}(C_{\widehat{\mu}})$ maps to the generator $(H-1)^{2k}$ of $\widetilde{K}(\mathbb{S}^{2k})$. The maps are the ones in the short exact sequence induced by the sequence of the pair $(C_{\widehat{\mu}}, \mathbb{S}^{2k})$.

for p prime, that is, $\psi^p(\eta) - \eta^p = p\nu$ for some $\nu \in K(X)$. These ring homomorphisms are called the **Adams operations**. Furthermore, because of their naturality condition, we have an operation

$$\psi^\ell : \widetilde{K}(X) \to \widetilde{K}(X)$$

given by restriction, since $\widetilde{K}(X)$ is the kernel of the homomorphism $K(X) \to K(x_0)$ induced by the inclusion of $x_0 \in X$ in X. For $X = \mathbb{S}^{2m}$, with a bit of work, one can prove that

$$\psi^\ell : \widetilde{K}(\mathbb{S}^{2m}) \simeq \mathbb{Z} \to \widetilde{K}(\mathbb{S}^{2m}) \simeq \mathbb{Z}$$

is multiplication by ℓ^m. Therefore, let $f : \mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ be a continuous map with Hopf invariant equal to plus or minus the unit. Let α and β be the elements of $\widetilde{K}(C_f)$ constructed in the process of defining the Hopf invariant of f. Since α is the image of a generator of $\widetilde{K}(\mathbb{S}^{4k})$,

$$\psi^\ell(\alpha) = \ell^{2k}\alpha.$$

Similarly,

$$\psi^\ell(\beta) = \ell^k\beta + \nu_\ell\alpha$$

for some $\nu_\ell \in \mathbb{Z}$. Therefore,

$$\psi^\ell\psi^m(\beta) = \psi^\ell(m^k\beta + \nu_m\alpha) = \ell^k m^k\beta + (\ell^{2k}\nu_m + m^k\nu_\ell)\alpha.$$

Since $\psi^\ell\psi^m = \psi^{\ell m} = \psi^m\psi^\ell$, the coefficient $\ell^{2k}\nu_m + m^k\nu_\ell$ of α is unchanged when ℓ and m are switched. This gives

$$\ell^{2k}\nu_m + m^k\nu_\ell = m^{2k}\nu_\ell + \ell^k\nu_m.$$

Equivalently,

$$\ell^k(\ell^k - 1)\nu_m = m^k(m^k - 1)\nu_\ell.$$

Because 2 is a prime number,

$$\psi^2(\beta) \equiv \beta^2 \pmod 2.$$

Since $\beta^2 = \pm\alpha$ because the Hopf invariant of f is plus or minus the unit, the formula $\psi^2(\beta) = 2^k\beta + \nu_2\alpha$ implies $\nu_2 \equiv \pm 1 \pmod 2$. Hence, ν_2 is odd. Moreover, for $\ell = 2$ and $m = 3$,

$$2^k(2^k - 1)\nu_3 = 3^k(3^k - 1)\nu_2.$$

Thence, $2^k \mid 3^k(3^k - 1)\nu_2$. But since 3^k and ν_2 are odd, 2^k must divide $3^k - 1$. Therefore, k must be 1, 2 or 4 by Lemma 7.1. □

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