Poincaré duality and the existence of exotic structures on n-spheres *

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Abstract. *Poincaré duality is a remarkable result in Algebraic Topology. It guarantees the existence of an isomorphism between the homology and cohomology groups of manifolds. We present a survey of the most general version of this result and its most important variations such as the Lefschetz duality and the Alexander duality. We consider an important application of these results in the study of the existence of exotic structures on n-spheres.*

Keywords – *Poincaré duality; Exotic sphere; Milnor Fibration.*

MSC2020 – *57P10*

1. Introduction

The concept of *duality* in mathematics emerged in the 19th century. After the term duality was used first in [1], Poincaré [2, 3, 4] used the Betti numbers of a closed orientable manifold to enunciate and to present a proof of the statement later known as the *Poincaré duality*, see Theorem 3.1.

A few years later, Alexander [5] stated and proved it with less restrictive hypotheses. Posteriorly, his study related to the Jordan-Brouwer separation Theorem culminated in the result known nowadays as the *Alexander duality*, see Theorem 3.8.

Finally, in 1926 Lefschetz [6], presented a version of Poincaré Duality in geometric topology, applying to a manifold with boundary. This version was called *Lefschetz duality*, see Theorem 3.7.

In the early 1930s, Alexander and Kolmogoroff, independently, found a definition of *cohomology*, which was announced at a conference in Moscow in 1935. They also suggested the existence of a product structure in cohomology that was studied in detail by Čech and Whitney. This latter determined the relationships and main properties of these

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products and, as a result, obtained different proofs of the Poincaré Duality, presented in the works [7, 8] and [9]. Thus, the advent of cohomology contributed decisively to the development of new tools and the evolution of the notations that gave rise to the modern statements and proofs of the Poincaré, Lefschetz, and Alexander dualities, which we present in this work.

The development of Poincaré Dualities as sophisticated tools stimulated the study and classification of topological spaces, sometimes equipped with a differentiable structure. In this context, naturally arose issues concerning bijectivity between classes of topological and differentiable structures. For instance:

Are there differentiable manifolds M and N equivalent in the topological sense, but not in the differentiable sense?

In 1956, Milnor presented the famous paper *"On manifolds homeomorphic to the* 7*-sphere"*, [10]. He presented a 7-dimensional smooth manifold homeomorphic to the standard 7-sphere with a non-equivalent differentiable structure.

The remarkable behavior of these spaces studied by Milnor lead to a specific terminology: *exotic spheres*. Thus, this term came to be used to denote a smooth manifold M which is homeomorphic but not diffeomorphic to the standard euclidean n-sphere.

In this way, many issues at the topological and geometric level came up, attributing the exotic spheres independence, becoming an original field of research, having been the subject, in many aspects, of several works published in the 1950s and in the following ones, among which we highlight [11, 12, 13].

The main goal here is to present a detailed survey of the dualities and the importance of these tools on the existence of exotic structures on n-spheres. In this sense, this paper divide as follows. In Section 2, important concepts and preliminary results are presented, such as the notion of direct systems, orientation, the fundamental class for a manifold, and the cohomology with compact support. All the results of section 2 are tools to build the "Poincaré homomorphism" and prove the Poincaré's duality in the first part of Section 3, Theorem 3.1. In the sequence, we state and prove the Lefschetz's Duality and Alexander's Duality, Theorem 3.7 and Theorem 3.8. In the last section of this work, we cover part of the path followed by Milnor in [14, Chapter 8], where Poincaré duality is used to ensure necessary and sufficient conditions for the existence of exotic structures on n-spheres.

2. Preliminaries

2.1. Direct Systems

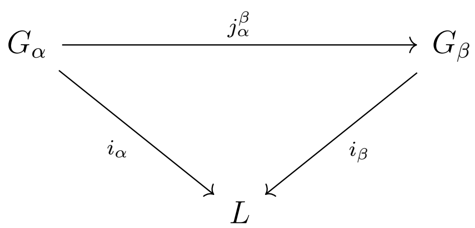
In this section, we present an essential notion in *Category Theory*: the *direct system of abelian groups*. It is fundamental in the proof of Poincaré Duality and the proof of its variations.

To define a direct system, we need a set with a notion of order to serve as a set of indexes. Let us consider the following definition.

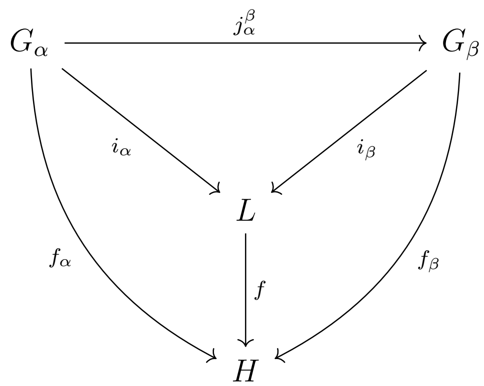
Definition 2.1. A direct set A is a partially ordered set $(A, \leq)$ such that for any $\alpha, \beta \in A$ there is an upper limit $\gamma \in A$ with $\alpha, \beta \leq \gamma$. A direct system of abelian groups is a collection $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in A}$ of abelian groups G_α indexed by a direct set A and homomorphisms $j_\alpha^\beta : G_\alpha \to G_\beta$, $\alpha \leq \beta$ such that $j_\beta^\gamma \circ j_\alpha^\beta = j_\alpha^\gamma$, $\alpha \leq \beta \leq \gamma$ and $j_\alpha^\alpha = 1$, $\alpha \in A$.

Definition 2.2. Let $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in A}$ be a direct system of abelian groups. An abelian group L is called the direct limit of $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in A}$ when there is a collection of homomorphisms $\{i_\alpha : G_\alpha \to L\}_{\alpha \in A}$ such that:

(i) For all $\alpha \leq \beta$ we have $i_\beta \circ j_\alpha^\beta = i_\alpha$, i.e., the following diagram is commutative:



(ii) (Universal property) If H is a abelian group and $\{f_\alpha : G_\alpha \to H\}_{\alpha \in A}$ is a family of morphisms such that $f_\beta \circ j_\alpha^\beta = f_\alpha$ with $\alpha \leq \beta$. Then, there exists a unique homomorphism $f : L \to H$ such that $f \circ i_\alpha = f_\alpha$, for all $\alpha \in A$. In other words, the following diagram is commutative:



The homomorphism $f : L \to H$ is denoted by $\{f_\alpha\} : L \to H$.

Proposition 2.3. *Let* $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in A}$ *be a direct system. If* $(L, i_\alpha)_{\alpha \in A}$ *and* $(L', i_\alpha)_{\alpha \in A}$ *are direct limits of* $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in A}$ *then* $L \cong L'$.

The previous proposition guarantees the uniqueness of the direct limit, up to isomorphisms. Consequently, one can denote this limit by

$$L := \varinjlim_{A} G_\alpha.$$

The next theorem describes the direct limit of any direct system of abelian groups.

Theorem 2.4. *Let* $\{G_\alpha, j_\alpha^\beta\}_{\alpha\in A}$ *be a direct system,* $i_\alpha : G_\alpha \to \bigoplus\limits_{\alpha\in A} G_\alpha$ *injections and* R *subgroup of* $\bigoplus\limits_{\alpha\in A} G_\alpha$ *generated by elements of the form* $i_\beta \circ j_\alpha^\beta(g) - i_\alpha(g)$, *with* $g \in G_\alpha$, *for all* $\alpha \leq \beta$, $\alpha, \beta \in A$. *Then*

$$\varinjlim_{A} G_\alpha \cong \frac{\bigoplus\limits_{\alpha\in A} G_\alpha}{R},$$

where the collection $\left\{ i_\alpha : G_\alpha \to \frac{\bigoplus_{\alpha\in A} G_\alpha}{R} \right\}$ *is induced by injections of the same nomenclature* $i_\alpha : G_\alpha \to \frac{\bigoplus_{\alpha\in A} G_\alpha}{R}$.

Definition 2.5. A morphism of direct systems

$$\{G_\alpha, j_\alpha^\beta\}_{\alpha\in A} \xrightarrow{\{\phi_\alpha\}} \{(G_\alpha)', (j_\alpha^\beta)'\}_{\alpha\in A}$$

is a collection of homomorphisms $\phi_\alpha : G_\alpha \to (G_\alpha)'$, $\alpha \in A$, satisfying $(j_\alpha^\beta)' \circ \phi_\alpha = \phi_\beta \circ j_\alpha^\beta$, i.e., the diagram below is commutative:

$$\begin{array}{ccccc} \alpha & & G_\alpha & \xrightarrow{\phi_\alpha} & (G_\alpha)' \\ \Big\downarrow{\scriptstyle (\alpha\leq\beta)} & & {\scriptstyle j_\alpha^\beta}\Big\downarrow & & \Big\downarrow{\scriptstyle (j_\alpha^\beta)'} \\ \beta & & G_\beta & \xrightarrow[\phi_\beta]{} & (G_\beta)' \end{array}$$

Whenever there is no risk of confusion, we use the simplest notation $\{\phi_\alpha\} : \{G_\alpha\} \to \{(G_\alpha)'\}$.

Corollary 2.6. *A morphism* $\{\phi_\alpha\} : \{G_\alpha\} \to \{(G_\alpha)'\}$ *of direct systems induces homomorphism*

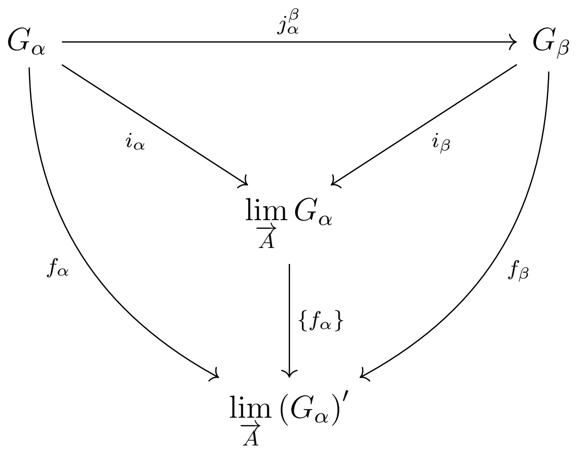
$$\varinjlim_{A} \phi_\alpha : \varinjlim_{A} G_\alpha \to \varinjlim_{A} (G_\alpha)'$$

such that $\varinjlim_{A} \phi_\alpha \circ i_\beta = i_\beta{}' \circ \phi_\beta$, *for all* $p \in A$.

Proof. For each $\alpha \in A$, define $f_\alpha := i_\alpha{}' \circ \phi_\alpha : G_\alpha \to \varinjlim_{A} (G_\alpha)'$. Given $\alpha \leq \beta$, $\alpha, \beta \in A$, one has

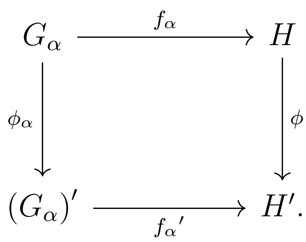
$$\begin{aligned} f_\beta \circ j_\alpha^\beta &= {i_\beta}' \circ \phi_\beta \circ j_\alpha^\beta \\ &= {i_\beta}' \circ (j_\alpha^\beta)' \circ \phi_\alpha \\ &= {i_\alpha}' \circ \phi_\alpha \\ &= f_\alpha, \end{aligned}$$

i.e., the following diagram is commutative:

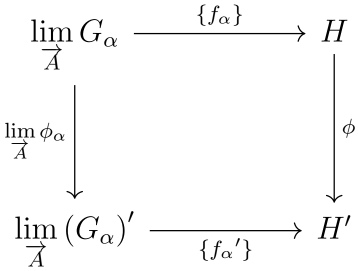


The universal property implies the existence of a unique homomorphism $\{f_\alpha\} : \varinjlim_A G_\alpha \to \varinjlim_A (G_\alpha)'$ such that $\{f_\alpha\} \circ i_\beta = f_\beta$, for all $\beta \in A$. Considering $\varinjlim_A \phi_\alpha := \{f_\alpha\}$, one has that $\varinjlim_A \phi_\alpha \circ i_\beta = {i_\beta}' \circ \phi_\beta$, for all $\beta \in A$. □

Corollary 2.7. *Assume that $\{G_\alpha\}$ and $\{(G_\alpha)'\}$ are direct systems and $f_\alpha : G_\alpha \to H$, ${f_\alpha}' : (G_\alpha)' \to H'$, $\alpha \in A$ are homomorphisms satisfying $f_\beta \circ j_\alpha^\beta = f_\alpha$, ${f_\beta}' \circ j_\alpha^{\beta\prime} = {f_\alpha}'$. Assume that $\{\phi_\alpha\} : \{G_\alpha\} \to \{(G_\alpha)'\}$ is a morphism of systems and $\phi : H \to H'$ homomorphism such that the diagram below commute for all $\alpha \in A$:*



Then, the diagram below is commutative.



Proof. Consider the following diagrams:

Diagram 1

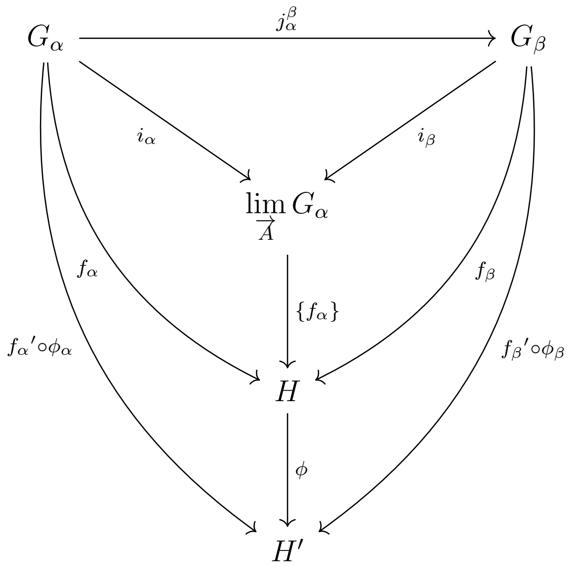
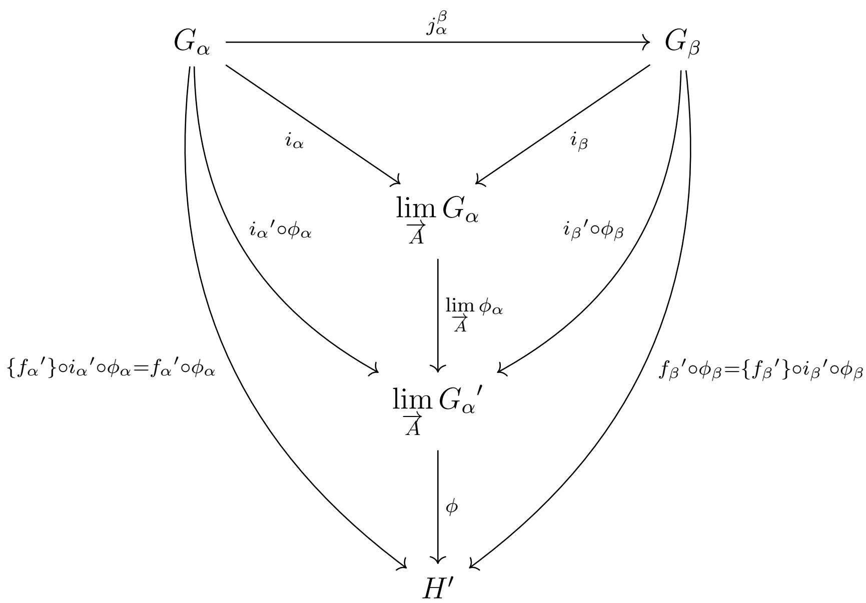


Diagram 2



In Diagram 1, we have $\{f_\alpha\}\circ i_\alpha = f_\alpha$ e $\phi\circ f_\alpha = {f_\alpha}'\circ\phi_\alpha$ which implies $(\phi\circ\{f_\alpha\})\circ i_\alpha = {f_\alpha}'\circ\phi_\alpha$, for all $\alpha\in A$. Thus, Diagram 2 and Corollary 2.6 imply that:

$$\varinjlim_{A} \phi_\alpha = {i_\alpha}'\circ\phi_\alpha.$$

In addition, for every α, we have $\{{f_\alpha}'\}\circ({i_\alpha}'\circ\phi_\alpha) = (\{{f_\alpha}'\}\circ{i_\alpha}')\circ\phi_\alpha = {f_\alpha}'\circ\phi_\alpha$. Therefore, $(\{{f_\alpha}'\}\circ\varinjlim_{A}\phi_\alpha)\circ {i_\alpha}' = \{{f_\alpha}'\}\circ{i_\alpha}'\circ\phi_\alpha = {f_\alpha}'\circ\phi_\alpha$. In words, the diagrams commute. Finally, the universal property for $\varinjlim_{A} G_\alpha$ implies $\phi\circ\{f_\alpha\} = \{{f_\alpha}'\}\circ\varinjlim_{A}\phi_\alpha$. □

The following proposition gives us a characterization of a direct limit that is calculable. More precisely, we have:

Proposition 2.8. *Let $\{G_\alpha, j_\alpha^\beta\}_{\alpha\in A}$ be direct system and consider on $\bigcup_{\alpha\in A} G_\alpha$ the following equivalence relation:*

- *Given $g\in G_\alpha$ and $g'\in G_\beta$, we have that $g\sim g'$ if and only if there is $\gamma\in A$ with $\gamma\geq\alpha, \gamma\geq\beta$ and $j_\alpha^\gamma(g) = j_\beta^\gamma(g')$, where the quotient*

$$\hat{G} := \frac{\bigcup_{\alpha\in A} G_\alpha}{\sim}$$

has a natural operation

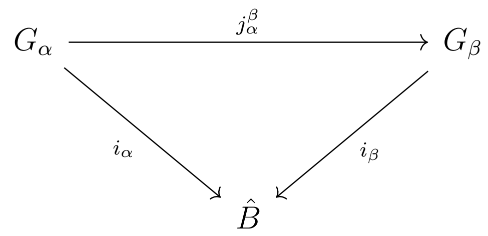
$$\begin{aligned} \oplus : \hat{G}\times\hat{G} &\to \hat{G} \\ (\{g\},\{g'\}) &\mapsto \{g\}\oplus\{g'\} := \{j_\alpha^\gamma(g) + j_\beta^\gamma(g')\}. \end{aligned}$$

If $i_\alpha : G_\alpha \to \hat{G}$ is defined by $i_\alpha(g) = \{g\}$, then $\{\hat{G}, i_\alpha\}$ is a direct limit of $\{G_\alpha, j_\alpha^\beta\}_{\alpha\in A}$. In particular, $\hat{G}\cong\varinjlim_{A} G_\alpha$.

Proof. Let $\alpha\leq\beta$ be with $\alpha,\beta\in A$. Given $g\in G_\alpha$, we have

$$i_\beta\circ j_\alpha^\beta(g) = i_\beta(j_\alpha^\beta) = \{j_\alpha^\beta(g)\} \quad \text{and} \quad i_\alpha(g) = \{g\}.$$

Since A is a direct set, there is $\gamma\in A$ such that $\gamma\geq\alpha$ and $\gamma\geq\beta$, which implies $\alpha\leq\beta\leq\gamma$. Now, once $\{G_\alpha\}_{\alpha\in A}$ is a direct system, then $j_\beta^\gamma\circ j_\alpha^\beta = j_\alpha^\gamma$ where we conclude that $j_\alpha^\gamma(g) = j_\beta^\gamma(j_\alpha^\beta(g))$. Thus, $j_\alpha^\beta(g)\sim g$ and we have equality $\{j_\alpha^\beta(g)\} = \{g\}$, which implies in $i_\beta\circ j_\alpha^\beta = i_\alpha$, that is, the diagram



is commutative. Let H be a abelian group and $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ a family of homomorphisms such that $f_\beta \circ j_\alpha^\beta = f_\alpha$, with $\alpha \leq \beta$. Define $f : \hat{G} \to H$ where $f(\{g\}) = f_\alpha(g)$, whenever $g \in G_\alpha$. In order to show that f is well defined, we consider $\{g\}, \{g'\} \in \hat{G}$ such that $\{g\} = \{g'\}$. We show that $f(\{g\}) = f(\{g'\})$. Suppose $g \in G_\alpha$ and $g' \in G_\beta$. Once $g \sim g'$ then there is $\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$ and $j_\alpha^\gamma(g) = j_\beta^\gamma(g')$. Then,

$$\begin{aligned} f_\alpha(g) &= f_\gamma \circ j_\alpha^\gamma(g) \\ &= f_\gamma(j_\alpha^\gamma(g)) \\ &= f_\gamma(j_\beta^\gamma(g')) \\ &= f_\beta(g'), \end{aligned}$$

i.e., $f_\alpha(g) = f_\beta(g')$. Therefore, $f(\{g\}) = f_\alpha(g) = f_\beta(g') = f(\{g'\})$. In addition, given $\alpha \in A$ and $g \in G_\alpha$, we have

$$f \circ i_\alpha(g) = f(\{g\}) = f_\alpha(g),$$

i.e., $f \circ i_\alpha = f_\alpha$, for all $\alpha \in A$. We claim that f is unique. Suppose there is $h : \hat{G} \to H$ such that $h \circ i_\alpha = f_\alpha$. Then, given $\alpha \in A$ and $g \in G_\alpha$, we have the sequence of equalities $h(\{g\}) = h \circ i_\alpha(g) = f_\alpha(g) = f(\{g\})$ which implies in $f = h$. Therefore, $(\hat{G}, i_\alpha)$ is a direct limit of the system $\{G_\alpha\}_{\alpha\in A}$. In particular, Proposition 2.3 gives us $\hat{G} \cong \varinjlim_{A} G_\alpha$. □

Proposition 2.9. *Suppose that $\{A_\alpha\}, \{B_\alpha\}$ and $\{C_\alpha\}$ are direct systems, $\{\phi_\alpha\} : \{A_\alpha\} \to \{B_\alpha\}$ and $\{\psi_\alpha\} : \{B_\alpha\} \to \{C_\alpha\}$ are morphisms of direct systems. If for each $\alpha \in A$, the sequence*

$$A_\alpha \xrightarrow{\phi_\alpha} B_\alpha \xrightarrow{\psi_\alpha} C_\alpha$$

is exact, then the sequence

$$\varinjlim_{A} A_\alpha \xrightarrow{\varinjlim_{A} \phi_\alpha} \varinjlim_{A} B_\alpha \xrightarrow{\varinjlim_{A} \psi_\alpha} \varinjlim_{A} C_\alpha$$

is also exact.

Proof. Consider the limit $(\hat{G}, i_\alpha) \cong \varinjlim_A G_\alpha$, given by Proposition 2.8. Note that $g \in G_\alpha$ is such that $i_\alpha(g) = O_{\hat{G}}$ if and only if

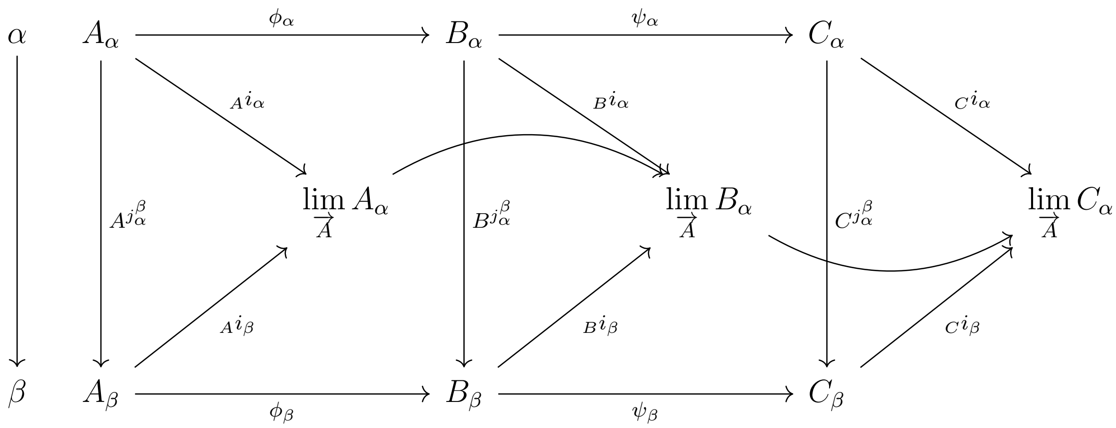
$$j_\alpha^\gamma(g) = O_{G_\gamma}, \tag{1}$$

for some $\gamma \geq \alpha$. In fact, if $i_\alpha(g) = 0$ then $\{g\} = O_{\hat{A}} = \{O_{G_\beta}\}$, hence there is $\gamma \in A$ such that $\gamma \geq \alpha, \gamma \geq \beta$ and $j_\alpha^\gamma(g) = j_\beta^\gamma(O_{G_\beta}) = O_{G_\gamma}$. On the other hand, equality (1) implies $j_\alpha^\gamma(g) = O_{G_\gamma} = j_\alpha^\gamma(O_{G_\alpha})$. Thus, we have that the equivalence $g \sim O_{G_\alpha}$ guarantees equality $\{g\} = \{O_{G_\alpha}\} = O_{\hat{G}}$ and therefore, $i_\alpha(g) = \{g\} = O_{\hat{G}}$. Consequently, $g \in \ker(i_\alpha)$ if and only if $g \in \ker(j_\alpha^\gamma)$, for some γ. In other words, we have $\ker(i_\alpha) = \ker(j_\alpha^\gamma)$, for some $\gamma \in A$.

Note that

$$\varinjlim_A \psi_\alpha \circ \varinjlim_A \phi_\alpha = \varinjlim_A(\psi_\alpha \circ \phi_\alpha) = 0.$$

In fact, be $\alpha \leq \beta$ in A, we have the following commutative diagram:



Of course, $\{\psi_\alpha \circ \phi_\alpha\} : \{A_\alpha\} \to \{C_\alpha\}$ is a morphism between direct systems , therefore induces a a unique homomorphism

$$\varinjlim_A(\psi_\alpha \circ \phi_\alpha) : \varinjlim_A A_\alpha \to \varinjlim_A C_\alpha$$

such that

$$\varinjlim_A(\psi_\alpha \circ \phi_\alpha) \circ_A i_\alpha =_C i_\alpha \circ (\psi_\alpha \circ \phi_\alpha). \tag{2}$$

On the other hand,

$$\begin{aligned}
\left(\lim_{\overrightarrow{A}}\psi_\alpha \circ \lim_{\overrightarrow{A}}\phi_\alpha\right)\circ_A i_\alpha &= \lim_{\overrightarrow{A}}\psi_\alpha \circ \left(\lim_{\overrightarrow{A}}\phi_\alpha \circ_A i_\alpha\right)\\
&= \lim_{\overrightarrow{A}}\psi_\alpha \circ ({}_B i_\alpha \circ \phi_\alpha)\\
&= \left(\lim_{\overrightarrow{A}}\psi_\alpha \circ_B i_\alpha\right)\circ \phi_\alpha\\
&= {}_C i_\alpha \circ (\psi_\alpha \circ \phi_\alpha). \qquad (3)
\end{aligned}$$

From the equations (2) and (3), we have

$$\lim_{\overrightarrow{A}}(\psi_\alpha \circ \phi_\alpha) = \lim_{\overrightarrow{A}}\psi_\alpha \circ \lim_{\overrightarrow{A}}\phi_\alpha.$$

Since the sequence

$$A_\alpha \xrightarrow{\phi_\alpha} B_\alpha \xrightarrow{\psi_\alpha} C_\alpha$$

is exact for all α, then $\psi_\alpha \circ \phi_\alpha = 0$ which implies equality $\lim_{\overrightarrow{A}}(\psi_\alpha \circ \phi_\alpha) = 0$. Therefore, $\lim_{\overrightarrow{A}}\psi_\alpha \circ \lim_{\overrightarrow{A}}\phi_\alpha = 0_{\hat{C}}$, and Im$(\lim_{\overrightarrow{A}}\phi_\alpha) \subseteq \ker(\lim_{\overrightarrow{A}}\psi_\alpha)$.

Finally, suppose $\{b\} \in \hat{B} \cong \lim_{\overrightarrow{A}} B_\alpha$ and it is such that $\lim_{\overrightarrow{A}}\psi(\{b\}) = 0_{\hat{C}}$ (i.e., $\{b\} \in \ker(\lim_{\overrightarrow{A}}\psi_\alpha)$). Considering $b \in B_\alpha$ and the commutative diagram below, for some $\beta \geq \alpha$:

$$\begin{array}{ccccc}
\alpha & & B_\alpha & \xrightarrow{\psi_\alpha} & C_\alpha\\
\Big\downarrow{\scriptstyle (\alpha\leq\beta)} & & \Big\downarrow{\scriptstyle {}_B j_\alpha^\beta} & & \Big\downarrow{\scriptstyle {}_C j_\alpha^\beta}\\
\beta & & B_\beta & \xrightarrow[\psi_\beta]{} & C_\beta,
\end{array}$$

we have

$$C^{j_\alpha^\beta} \circ \psi_\alpha(b) = \psi_\beta \circ B^{j_\alpha^\beta}(b). \qquad (4)$$

Once

$$\begin{aligned}
{}_B i_\alpha : B_\alpha &\to \hat{B} \cong \lim_{\overrightarrow{A}} B_\alpha,\\
b &\mapsto \{b\},
\end{aligned}$$

one has

$$\begin{aligned} 0_{\hat{C}} &= \varinjlim_{A} \psi_\alpha(\{b\}) \\ &= \left(\varinjlim_{A} \psi_\alpha\right) \circ_B i_\alpha(b) \\ &= {}_C i_\alpha(\psi_\alpha(b)). \end{aligned}$$

Hence, we have that $\psi_\alpha(b) \in \ker({}_C i_\alpha)$. By equation (1) we have $C^{j_\alpha^\beta}(\psi_\alpha(b)) = 0_{C_\alpha}$. The previous equality and the equation (4) implies that $\psi_\beta(B^{j_\alpha^\beta}(b)) = 0_{C_\alpha}$. Now, the sequence

$$A_\beta \xrightarrow{\ \phi_\beta\ } B_\beta \xrightarrow{\ \psi_\beta\ } C_\beta$$

is exact, and since $B^{j_\alpha^\beta}(b) \in \ker(\psi_\beta) = \mathrm{Im}(\phi_\beta)$ then there is $a \in A_\beta$ such that $\phi_\beta(a) = B^{j_\alpha^\beta}(b)$. Therefore,

$$\begin{aligned} \varinjlim_{A} \phi_\alpha(\{a\}) &= \varinjlim_{A} \phi_\alpha \circ_A i_\beta(a) \\ &= {}_B i_\beta \circ \phi_\beta(a) \\ &= {}_B i_\beta \circ B^{j_\alpha^\beta}(b) \\ &= {}_B i_\alpha(b) \\ &= \{b\}, \end{aligned}$$

i.e., $\varinjlim_{A} \phi_\alpha(\{a\}) = \{b\}$ which implies that $\{b\} \in \mathrm{Im}(\varinjlim_{A} \phi_\alpha)$. Thus, the inclusion

$$\ker\left(\varinjlim_{A} \psi_\alpha\right) \subseteq \mathrm{Im}\left(\varinjlim_{A} \phi_\alpha\right),$$

implies equality

$$\ker\left(\varinjlim_{A} \psi_\alpha\right) = \mathrm{Im}\left(\varinjlim_{A} \phi_\alpha\right).$$

Thus, we conclude that the sequence

$$\varinjlim_{A} A_\alpha \xrightarrow{\ \varinjlim_{A} \phi_\alpha\ } \varinjlim_{A} B_\alpha \xrightarrow{\ \varinjlim_{A} \psi_\alpha\ } \varinjlim_{A} C_\alpha$$

is exact. □

Corollary 2.10. *If $\{\phi_\alpha\} : \{G_\alpha\} \to \{(G_\alpha)'\}$ is a morphism of direct systems such that each ϕ_α is a monomorphism (resp. epimorphism, resp. isomorphism) then the same is true for $\varinjlim_{A} \phi_\alpha$.*

Proof. The proof follows directly from the Proposition 2.9. □

Proposition 2.11. *Assume that $(I, \leq)$ is a direct set of indices and $\{G_\alpha, j_\alpha^\beta\}_{\alpha \in I}$ is a direct system of abelian groups. Suppose that $J \subset I$ is such that for each $\alpha \in I$ there is $\beta \in J$ with $\alpha \leq \beta$. Then $\varinjlim_{\alpha \in I} G_\alpha = \varinjlim_{\alpha \in J} G_\alpha$. In particular, if I has a maximal element γ (i.e., given $\alpha \in I, \alpha \leq \gamma$), taking $J = \{\gamma\}$, we have*

$$\varinjlim_{\alpha \in I} G_\alpha = G_\gamma.$$

Proposition 2.12. *Let X be a space and $\{X_\alpha\}_{\alpha \in A}$ a collection of subspaces of X ordered by the inclusion relation such that for each compact set $K \subset X$, there is $\gamma \in A$ with $K \subset X_\gamma$. Be $j_\alpha^\beta : X_\alpha \to X_\beta$ and $i_\alpha : X_\alpha \to X$ the inclusions $(\alpha \leq \beta)$. Consequently, fixed $p \in \mathbb{Z}^*$ and G an abelian group, the abelian groups $H_p(X_\alpha, G)$ and homomorphisms $j_{\alpha_*}^\beta : H_p(X_\alpha, G) \to H_p(X_\beta, G)$ form a direct system of abelian groups*

$$\left\{H_p(X_\alpha, G), j_{\alpha_*}^\beta\right\}_{\alpha \in A}.$$

In addition, the induced homomorphism

$$\{i_{\alpha_*}\} : \varinjlim_{\alpha} H_p(X_\alpha, G) \to H_p(X, G)$$

is an isomorphism, for all $p \in \mathbb{Z}^*$.

2.2. Orientable manifolds and the fundamental class

Definition 2.13. Given a space X and a point $x \in X$, the local homology group of x in X are the groups $H_n(X, X \setminus \{x\})$.

Note that given an open neighborhood U_x of x, the Excision Theorem guarantees the existence of an isomorphism $H_n(X, X \setminus \{x\}) \cong H_n(U_x, U_x \setminus \{x\})$. Thus, the local homology group depends only on the local topology of X close to x.

Also, if $f : X \to Y$ is a homeomorphism then it induces an isomorphism $f_*^n : H_n(X, X \setminus \{x\}) \to H_n(Y, Y \setminus \{f(x)\})$, for all $x \in X$ and for all n. Consequently, groups of local homology are used to tell us when spaces are not locally homeomorphic at certain points.

Definition 2.14. A n-dimensional manifold (or n-manifold) is a Hausdorff space M where each point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Note that the Excision Theorem guarantees that $H_i(M, M \setminus \{x\}, \mathbb{Z}) \cong H_i(U_x, U_x \setminus \{x\})$. Since $U_x \cong \mathbb{R}^n$, we have $H_i(U_x, U_x \setminus \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}, \mathbb{Z})$. Now, considering the sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ and the fact that $\mathbb{R}^n$ is contractible, we have that

$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\} \cong H_{i-1}(\mathbb{R}^n \setminus \{0\}, \mathbb{Z})$. Finally, since $\mathbb{R}^n \setminus \{0\} \cong S^{n-1}$, we conclude that $H_i(M, M \setminus \{x\}, \mathbb{Z}) \cong H_{i-1}(S^{n-1}, \mathbb{Z})$. Thus, the dimension of M is intrinsically characterized by the fact that that for $x \in M$, the local homology group $H_i(M, M \setminus \{x\}, \mathbb{Z})$ is nonzero only when $i = n$.

Definition 2.15. A n-manifold M is said to be closed if it is compact and without boundary.

The Poincaré duality states that for a closed n-manifold and orientable there are isomorphisms $H^n(M, \mathbb{Z}) \cong H_{n-p}(M, \mathbb{Z})$, for all $p \in \mathbb{Z}^+$.

By convention, cohomology and homology groups of negative dimension are zero. So a duality statement includes the fact that any non-trivial cohomology and homology groups of M occur for dimension between 0 and n.

In what follows, we begin the discussion necessary to present the definition of *orientation* for a n-manifold M.

Recall that $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, where S^{n-1} is the sphere with center in x. With this in mind, we present the next definition.

Definition 2.16. An orientation of $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is a choice of the generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$.

Let $p : \mathbb{R}^n \to \mathbb{R}^n$ be a rotation, $r : \mathbb{R}^n \to \mathbb{R}^n$ a reflection and $x \in X$. Consider the induced homomorphisms $p_*^n, r_*^n : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$. If $\alpha \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ is a generator, we can show that $p_*^n(\alpha) = \alpha$ and $r_*^n(\alpha) = -\alpha$, i.e., the orientation α of $\mathbb{R}^n$ at a point x is preserved by rotations and inverted by reflections, just like $\mathbb{R}^2$ with the notions of clockwise and counterclockwise.

Remark 2.17. Set arbitrarily $x \in \mathbb{R}^n$. Note that the choice of a generator $\alpha_x \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ determines, for each $y \in \mathbb{R}^n$, the choice of a generator $\alpha_y \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$, via canonical isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$, where B is a closed ball containing x and y. In other words, an orientation of $\mathbb{R}^n$ at a point x, determines an orientation at each other point $y \in \mathbb{R}^n$.

Definition 2.18. A local orientation of M at a point $x \in M$ is a choice of a generator μ_x of the cyclic group $H_n(M, M \setminus \{x\})$.

Notation 2.19. *Next, we use the following simpler notations:*

$$H_n(X, X \setminus A) = H_n(X \mid A),$$

or, more generally

$$H_n(X, X \setminus A; G) = H_n(X \mid A; G).$$

Finally, if M is a n-manifold and $x \in M$, we denote $H_n(M, M \setminus \{x\}) = H_n(M \mid x)$.

By Excision Theorem, $H_n(X \mid A)$ depends only on a closure neighborhood of the $\overline{A}$ from A in X. Thus, it makes sense to see $H_n(X \mid A)$ as a local homology of X in A.

We now define a global orientation in a n-manifold M, which should be a consistent choice of an orientation at each point in the manifold.

Definition 2.20. A $\mathbb{Z}$-orientation (or simply, *orientation*) on a n-dimensional manifold M is a function

$$\begin{aligned} \theta_M \quad : \quad & M \to \bigcup_{x\in M} H_n(M \mid x) \\ & x \mapsto \theta_M(x) = \mu_x \end{aligned}$$

that associates with each $x \in M$ a local orientation $\mu_x \in H_n(M \mid x)$ satisfying the following condition, known as *Local Consistency Condition*, or simply, LC:

LC: For each $x \in M$, there is a neighborhood $U_{\mathbb{R}^n} \subset M$ homeomorphic to $\mathbb{R}^n$, via homeomorphism $\phi : U_{\mathbb{R}^n} \to \mathbb{R}^n$, and an open set $B_x \subset U_{\mathbb{R}^n}$, homeomorphic to an open ball $B_{\phi(x)} \subset \mathbb{R}^n$ of center $\phi(x)$ and finite radius, such that for each $y \in B_x$, the orientation $\mu_y \in H_n(M \mid y)$ is an image of a generator $\mu_{B_x} \in H_n(M \mid B_x)$ under the natural map $\psi_y : H_n(M \mid B_x) \to H_n(M \mid y)$.

When there is an orientation for M, we say that M is *orientable*.

Let M be a orientable n-manifold and $x \in M$, $\phi : U_{\mathbb{R}^n} \to \mathbb{R}^n$ a homeomorphism and $B_x \subset U_{\mathbb{R}^n}$ homeomorph to an open ball $B_{\phi(x)} \subset \mathbb{R}^n$, as described above. Next, we use the following commutative diagram to ensure that the Local Consistency Condition holds:

$$\begin{array}{ccccccccc}
\mu_x & & H_n(M \mid x) & \xleftarrow{\psi_x} & H_n(M \mid B_x) & \xrightarrow{\psi_y} & H_n(M \mid y) & & \mu_y \\
\Big\downarrow & & \text{Excision}\Big\downarrow\cong & & \Big| & & \cong\Big\downarrow\text{Excision} & & \Big\downarrow \\
g_1 & & H_n(U_{\mathbb{R}^n} \mid x) & & \cong\Big\downarrow(\phi_{|B_x})^n_* & & H_n(U_{\mathbb{R}^n} \mid y) & & g_2 \\
\Big\downarrow & & \phi^n_*\Big\downarrow\cong & & \Big| & & \cong\Big\downarrow\phi^n_* & & \Big\downarrow \\
g_1(\mu_x) & \in & H_n(\mathbb{R}^n \mid \phi(x)) & \xrightarrow[\cong]{G_1} & H_n(\mathbb{R}^n \mid B_{\phi(x)}) & \xrightarrow[\cong]{G_2} & H_n(\mathbb{R}^n \mid \phi(y)) & \ni & g_2(\mu_y)
\end{array}$$

Once $x \in B_x$, we have that $\mu_x = \psi_x(\mu_{B_x})$. Also, given $y \in B_x$ we have $\mu_y = \psi_y(\mu_{B_x})$. These orientations in x and y induce orientations $g_1(\mu_x)$ and $g_2(\mu_y)$ from $\mathbb{R}^n$ in $\phi(x)$ and $\phi(y)$ via the isomorphisms g_1 and g_2, respectively.

Note that $\phi(x), \phi(y) \in B_{\phi(y)}$ and by Remark 2.17, $G_2(G_1(g_1(\mu_x))) \in H_n(\mathbb{R}^n \mid \phi(y))$ is an orientation of $\mathbb{R}^n$ in $\phi(y)$. Since the diagram above is commutative, we have $\mu_x = \psi_x(\mu_{B_x}) = g_1^{-1} \circ G_1^{-1}((\phi_{|B_x})^n_*(\mu_{B_x}))$ and consequently,

$$G_1(g_1(\mu_x)) = (\phi_{|B_x})^n_*(\mu_{B_x}). \tag{5}$$

Moreover, $\mu_y = \psi_y(\mu_{B_x}) = g_2^{-1}(G_2((\phi_{|B_x})^n_*(\mu_{B_x}))$, therefore, $g_2(\mu_y) = G_2((\phi_{|B_x})^n_*(\mu_{B_x})$. So, the last equation and (5) imply that

$$g_2(\mu_y) = G_2(G_1(g_1(\mu_x))),$$

i.e., the orientation $G_2(G_1(g_1(\mu_x)))$ of $\mathbb{R}^n$ in $\phi(y)$ induced by the orientation of $\mathbb{R}^n$ in $\phi(x)$ is compatible with the orientation of $\mathbb{R}^n$ in $\phi(y)$ induced by orientation μ_y of M in y induced by g_2.

In other words, if M is orientable then for each $x \in M$, there is a neighborhood B_x of x in M such that for each $y \in B_x$ the orientation $g_2(\mu_y)$ is induced by the orientation $g_1(\mu_x)$, and that is exactly the meaning of the term "*local consistency*".

Example 2.21. If M is simply connected, or more generally, if $\pi_1(M)$ don't have index two subgroups, then M is orientable. See [15, Proposition 3.25]

We can generalize the definition of orientation by replacing the ring $\mathbb{Z}$ with any commutative ring with unit R.

Definition 2.22. An R-orientation in M is a function

$$\begin{aligned} \theta^R_M \quad : \quad & M \to \bigcup_{x\in M} H_n(M \mid x, R) \\ & x \mapsto \theta^R_M(x) = \mu^R_x \end{aligned}$$

that associates each x a generator μ^R_x of $H_n(M \mid x, R) \cong R$ satisfying a corresponding local consistency condition, where μ^R_x is an element R such that $R \cdot \mu^R_x = R$.

Since we assume that R has an identity element $1_R \in R$ then there is $b \in R$ such that $1_R = \mu^R_x b$. Therefore, μ^R_x is an invertible element of R.

It is important to note that every orientable n-manifold M is also R-orientable. Otherwise, if M is non-orientable, then it is R-orientable if and only if the ring R contains a unity of order 2. An immediate consequence of this fact is that every manifold is $\mathbb{Z}_2$-orientable. Consequently, the most important R-orientation cases occur when $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$.

We end this section by defining the *fundamental class for a manifold M*, which is decisive for the Poincaré Duality. Before that, we consider the following lemma.

Lemma 2.23. *[15] Let M be a n-manifold and $A \subset M$ a compact subset. Then:*

(a) $H_p(M \mid A, R) = 0$, for all $p > n$. Also, given $[\varphi] \in H_n(M \mid A, R)$ we have $[\varphi] = 0$ in $H_n(M \mid A, R)$ if and only if, considering the homomorphisms

$$\psi_x : H_n(M \mid A, R) \to H_n(M \mid x, R),$$

we have that $\psi_x([\varphi]) = 0$, for all $x \in A$.

(b) There is a single class $[\alpha_A] \in H_n(M \mid A, R)$ such that $\psi_x([\alpha_A]) = \theta_M^R(x)$, for each $x \in A$.

Definition 2.24. Let M be a n-manifold. For each $x \in M$, consider the homomorphisms $\psi_x : H_n(M, R) \to H_n(M \mid x, R)$. An element $[M] \in H_n(M, R)$ such that $\psi_x([M]) = \mu_x$ is a generator of $H_n(M \mid x, R)$ is called a fundamental class for M with coefficients in R.

Theorem 2.25. *[15] Let M be a connected closed n-manifold. If M is R-orientable then the homomorphism $\psi_x : H_n(M, R) \to H_n(M \mid x, R) \cong R$ is an isomorphism, for every $x \in M$.*

Consequently, if M is a connected closed n-manifold and R-orientable, Theorem 2.25 guarantees the existence of a fundamental class $[M] \in H_n(M, R)$. On the other hand, in [15] one can see in the case where M is a n-manifold without boundary, if there is a fundamental class $[M] \in H_n(M, R)$, then M is compact and R-orientable.

2.3. Cohomology with compact support

In order to make clear both the statement and the proof of Poincaré's Duality in its most general version, we need to consider cohomology groups with compact support.

We begin by considering X a topological space and defining the set

$$A = \{K \subset X \mid K \text{ is a compact subset of } X\}.$$

The inclusion relation determines about A a partial order relationship. Since the union of two compacts sets is compact, for any $K, L \in A$, one has that $T := K \cup L \in A$, $K \leq T$ and $L \leq T$. Therefore, $(A, \leq)$ is a direct set. Consider inclusion $j_K^L : K \to L$. Since $K \leq L$, we have that $X \setminus L \subset X \setminus K$. Fixed $p \in \mathbb{Z}^*$ and G an abelian group, we can associate with each inclusion j_K^L, the natural homomorphism of abelian groups ${j_K^L}^* : H^p(X, X \setminus K; G) \to H^p(X, X \setminus L; G)$ (induced by $j : (X, X \setminus L) \to (X, X \setminus K)$). It is possible to show that ${j_L^T}^* \circ {j_K^L}^* = {j_K^T}^*$, $K \leq L \leq T$ and ${j_K^K}^* = 1$, for all $K \in A$. Thus, $\{H^p(X, X \setminus P; G), {j_K^L}^*\}_{K \in A}$ is a direct system of abelian groups.

By Excision Lemma, the group $H^p(X, X \setminus K; G)$, where K is a compact set, depends only on a neighborhood of K in X (assuming that X is Hausdorff, we have that

K is a closed set). Let us shorten the notation by writing

$$H^p(X, X \setminus K; G) = H^p(X \mid K; G).$$

Definition 2.26. We define

$$H_c^p(X; G) = \varinjlim_{K} H^p(X \mid K; G)$$

as the cohomology group with compact support. Notice the similar notation used for local homology. One can think of homology groups with compact support as the limit of these "local cohomology groups in compact subsets´´.

Remark 2.27. (a) The Proposition 2.8 provides us with the following alternative description for the cohomology group with compact support:

$$H_c^p(X, G) = \frac{\bigcup_{K \in A} H^p(X \mid K; G)}{\sim}.$$

Therefore, for each element $\varphi \in H_c^p(X, G)$, there are a compact set $K \in A$ and a element $\alpha_{\varphi,K} \in H^p(X \mid K; G)$ such that $\varphi = \{\alpha_{\varphi,K}\}$, where "{ }" represents the class of the element $\alpha_{\varphi,K}$ according to the equivalence relation "$\sim$" given by Proposition 2.8.

(b) Proposition 2.11 implies that for compact spaces X, fixed $p \in \mathbb{Z}$ non-zero and G an abelian group, we have that $H_c^p(X, G) = H^p(X, G)$.

The next proposition is useful to prove a weaker version of Poincaré Duality (see Proposition 3.5).

Proposition 2.28. *Let $X = \mathbb{R}^n$ and $G = \mathbb{Z}$. For any $n \in \mathbb{N}$, we have*

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{Z}; & \textit{if } p = n \\ 0; & \textit{if } p \neq n \end{cases}.$$

Proof. We know that $H_c^p(\mathbb{R}^n) = \varinjlim_{K \in A} H^p(\mathbb{R}^n \mid K)$, where $A = \{K \subset \mathbb{R}^n : K \text{ is compact}\}$. Consider the set

$$A' := \{B[0, l] \subset \mathbb{R}^n : l \in \mathbb{N}, B[0, l] \text{ is the closed ball with center at origin and radius } l\}.$$

Note that $A' \subset A$. Also, given $K \in A$, there is $l_K \in \mathbb{N}$ such that $B[0, l_K] \in A'$ and $K \subset B[0, l_k]$. Thus,

$$\varinjlim_{B[0,l] \in A'} H^p(\mathbb{R}^n \mid B[0, l]) = \varinjlim_{K \in A} H^p(\mathbb{R}^n \mid K).$$

Therefore,

$$H_c^p(\mathbb{R}^p) = \varinjlim_{A'} H^p(\mathbb{R}^n \mid B[0,l]) = \varinjlim_{l\in\mathbb{N}} H^p(\mathbb{R}^n \mid B[0,l]). \tag{6}$$

On the other hand, we have $H^p(\mathbb{R}^n \mid B[0,l]) \cong H^p(\mathbb{R}^n \mid \{x_0\}) \cong H^{p-1}(\mathbb{R}^n \mid \{x_0\}) = H^{p-1}(S^{n-1})$. Consequently,

$$H^p(\mathbb{R}^n \mid B[0,l]) \cong \begin{cases} \mathbb{Z}; \text{ if } p = n \\ 0; \text{ if } p \neq n \end{cases}. \tag{7}$$

Hence, (7) and (6) imply

$$H_c^p(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}; \text{ if } p = n \\ 0; \text{ if } p \neq n \end{cases}.$$

□

3. The Poincaré Duality and its variations

In this section, we use the cap product to state the Poincaré Duality as it has been done in [15].

3.1. Poincaré Duality

Let R be a commutative ring with unit, M a n-manifold possibly non-compact, connected and R-orientable. We start this section by building a dual homomorphism between the cohomology groups with compact support $H_c^p(M;R)$ and the homology groups $H_{n-p}(M;R)$ for all $p \in \mathbb{Z}^+$, as follows:

(1) For compact subsets K, L of M, with $K \subset L$, we consider the inclusion $i_K^L : K \hookrightarrow L$ and the associated homomorphisms i_{K*}^L and i_K^{L*}, induced by inclusion $i : (M, M \setminus L) \to (M, M \setminus K)$, we get the following diagram:

$$\begin{array}{ccc} H_n(M \mid L,R) \times H^p(M \mid L,R) & \xrightarrow{\quad\frown\quad} & H_{n-p}(M,R) \\ \downarrow i_{K*}^L \qquad\qquad \uparrow i_K^{L*} & & \downarrow \mathbb{1}_{H_{n-p}(M,R)} \\ H_n(M \mid K,R) \times H^p(M \mid K,R) & \xrightarrow{\quad\frown\quad} & H_{n-p}(M,R) \end{array}$$

(2) By item (b) of Lemma 2.23, there are unique elements $\mu_K \in H_n(M \mid K;R)$ e $\mu_L \in H_n(M \mid L;R)$ such that $\psi_x^K(\mu_K) = \mu_x$ and $\psi_y^L(\mu_L) = \mu_y$, for any $x \in K$ and $y \in L$, where $\psi_x^K : H_n(M \mid K;R) \to H_n(M \mid x,R)$, $\psi_y^L : H_n(M \mid L;R) \to H_n(M \mid y,R)$, $\mu_x \in H_n(M \mid x,R)$ and $\mu_y \in H_n(M \mid y,R)$ are

orientations of M at each point $x \in K$ e $y \in L$, respectively. By uniqueness, we have $i^L_{K*}(\mu_L) = \mu_K$.

(3) Given $\alpha \in H^p(M \mid K; R)$, it is possible to show that

$$i^L_{K*}(\mu_L) \frown \alpha = \mu_L \frown {i^L_K}^*(\alpha).$$

Therefore, once $i^L_{K*}(\mu_L) = \mu_K$ we get

$$\mu_K \frown \alpha = \mu_L \frown {i^L_K}^*(\alpha) \in H_{n-p}(M; R),$$

for all $\alpha \in H^p(M \mid K; R)$

(4) Let $A := \{L \subset M \,|\, L \text{ is compact}\}$. We consider for each $K \in A$ the homomorphism

$$\begin{array}{rccc} D_K : & H^p(M \mid K; R) & \to & H_{n-p}(M; R) \\ & \alpha & \mapsto & \mu_K \frown \alpha \end{array}$$

We define $G_K := H_{n-p}(M; R)$, for all $K \in A$ and $\Theta^L_K := \mathbb{1}_{H_{n-p}(M;R)}$, for all $K, L \in A$ with $K \subset L$. We know that $\{G_K, \Theta^L_K\}_{K \in A}$ is a direct system of abelian groups and

$$H_{n-p}(M, R) = \varinjlim_{K} G_K.$$

Now, we consider the map

$$\{D_K\} : \left\{H^p(M \mid K, R); {i^L_K}^*\right\}_{K\in A} \to \left\{G_K, \Theta^L_K\right\}_{K\in A}$$

as being the collection of homomorphisms D_K as defined above.

(5) Let us show that $\{D_K\}$ is a homomorphism of direct systems. In fact, for all $\alpha \in H^p(M \mid K, R)$, we have that

$$\Theta^L_K(D_K(\alpha)) = \Theta^L_K(\mu_K \frown \alpha) = \mu_K \frown \alpha = \mu_K \frown {i^L_K}^*(\alpha) = D_L\left({i^L_K}^*(\alpha)\right),$$

that is, the diagram

$$\begin{array}{ccccc} K & & H^p(M \mid K, R) & \xrightarrow{D_K} & G_K = H_{n-p}(M, R) \\ \Big\downarrow {\scriptstyle K\subset L} & & \Big\downarrow {\scriptstyle {i^L_K}^*} & & \Big\downarrow {\scriptstyle \Theta^L_K} \\ L & & H^p(M \mid L, R) & \xrightarrow[D_L]{} & G_L = H_{n-p}(M, R) \end{array}$$

is commutative and $\Theta^L_K \circ D_K = D_L \circ {i^L_K}^*$.

(6) Corollary 2.6 guarantees that homomorphism of direct systems $\{D_K\}$ induces a

homomorphism

$$\lim_{\overrightarrow{K}} D_K : \lim_{\overrightarrow{K}} H^p(M \mid K; R) \to \lim_{\overrightarrow{K}} G_K. \tag{8}$$

Once $\lim_{\overrightarrow{K}} H^p(M \mid K; R) = H^p_c(M; R)$, after we denote $D_M = \lim_{\overrightarrow{K}} D_K$, we get from (8) the desired dual homomorphism:

$$D_M : H^p_c(M; R) \to H_{n-p}(M; R).$$

Since $H^p_c(M; R)$ is a direct limit, Remark 2.27 and Corollary 2.6 ensure the existence of a collection of homomorphisms

$$\begin{array}{rccc} \eta_K : & H^p(M \mid K; R) & \to & H^p_c(M; R) \\ & \alpha & \mapsto & \{\alpha\}, \end{array}$$

satisfying $D_M \circ \eta_K = D_K$, for all $K \in A$. Finally, for all $\varphi \in H^p_c(M; R)$, we have that

$$D_M(\varphi) = D_M(\{\alpha_{\varphi,K}\}) = D_M(\eta_K(\alpha_{\varphi,K})) = D_K(\alpha_{\varphi,K}) = \mu_K \frown \alpha_{\varphi,K}.$$

Concerning the dual map, from now on, we denote $D_M(\varphi) = \mu_K \frown \varphi$.

Now, we are ready to enunciate Poincaré's Duality Theorem:

Theorem 3.1 (Poincaré Duality)**.** *Let M be an R-orientable n-manifold. The dual homomorphism $D_M : H^p_c(M, R) \to H_{n-p}(M, R)$ is an isomorphism for all $p \in \mathbb{Z}^+$.*

To demonstrate Theorem 3.1, we need some preliminary results. We start with the following technical lemma:

Lemma 3.2. *[16] If M is the union of two open sets U e V, i.e., $M = U \cup V$, then the diagram below:*

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^p_c(U\cap V) & \longrightarrow & H^p_c(U)\oplus H^p_c(V) & \longrightarrow & H^p_c(M) & \longrightarrow & H^{p+1}_c(U\cap V) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U\oplus -D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} & \\ \cdots \longrightarrow & H_{n-p}(U\cap V) & \longrightarrow & H_{n-p}(U)\oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(M) & \longrightarrow & H_{n-p-1}(U\cap V) & \longrightarrow \cdots \end{array}$$

is commutative and its horizontal lines are exact.

Proof. Let $K \subset U$ and $L \subset V$ compact sets in U and V, respectively. Keeping the Theorem 3.9 (b) in mind, we take $X = A = B = M$, $C = M \setminus K$, $D = M \setminus L$ and $Y = C \cup D$. Thus, $Y = M \setminus (K \cap L)$, $C \cap D = M \setminus (K \cup L)$, $X = M$, $A \cup B = X$ and $A \cap B = M$. These sets give rise to Mayer-Vietoris sequences, where the first is the relative sequence for cohomology

$$\cdots \longrightarrow H^p(M \mid K \cap L) \xrightarrow{\Psi_p} H^p(M \mid K) \oplus H^p(M \mid L) \xrightarrow{\Phi_p} H^p(M \mid K \cup L) \xrightarrow{\Delta_p} \cdots$$

The second one is a Mayer-Vietoris sequence for homology:

$$\cdots \longrightarrow H_{n-p}(U \cup V) \xrightarrow{\psi_{n-p}} H_{n-p}(U) \oplus H_{n-p}(V) \xrightarrow{\phi_{n-p}} H_{n-p}(M) \xrightarrow{\Delta_p} \cdots$$

Now we take $X = M$, $A = M \setminus (K \cap L)$ and $U = (M \setminus V) \cup (M \setminus U)$ and consequently, (X, A) is such that $\bar{U} \subset \text{int}A$, $X \setminus U = U \cap V$ and $A \setminus U = (U \cap V) \setminus (K \cap L)$. Thus $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$\alpha : H^p(M \mid K \cap L) \longrightarrow H^p(U \cap V \mid K \cap L),$$

and making other suitable choices, we obtain the isomorphism

$$\beta : H^p(M \mid K) \oplus H^p(M \mid L) \longrightarrow H^p(U \mid K) \oplus H^p(V \mid L).$$

Now, we can consider the dual homomorphisms induced by the cap products: $\mu_{K\cap L} \frown \_$, $\mu_K \frown \_ \oplus -\mu_L \frown \_$ and $\mu_{KU} \frown \_$, i.e.,

$$\begin{aligned} D_{K\cap L} : H^p(U \cap V \mid K \cap L) &\longrightarrow H_{n-p}(U \cap V) \\ \varphi &\longmapsto \mu_{K\cap L} \frown \varphi; \end{aligned}$$

$$\begin{aligned} D_K \oplus (-D_L) : H^p(U \mid K) \oplus H^p(V \mid L) &\longrightarrow H_{n-p}(U) \oplus H_{n-p}(V) \\ (\varphi, \psi) &\longmapsto (\mu_K \frown \varphi) \oplus (-\mu_L \frown \psi); \end{aligned}$$

$$\begin{aligned} D_{K\cup L} : H^p(M \mid K \cup L) &\longrightarrow H_{n-p}(M) \\ \varphi &\longmapsto \mu_{K\cup U} \frown \varphi. \end{aligned}$$

We can see that the below diagram is commutative:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^p(M \mid K \cap L) & \xrightarrow{\Psi_p} & H^p(M \mid K) \oplus H^p(M \mid L) & \xrightarrow{\Phi_p} & H^p(M \mid K \cup L) & \xrightarrow{\Delta_p} & H^{p+1}(M \mid K \cap L) & \longrightarrow \cdots \\
& \cong \downarrow \alpha & & \cong \downarrow \beta & & & & \downarrow \cong & \\
& H^p(U \cap V \mid K \cap L) & \longrightarrow & H^p(U \mid K) \oplus H^p(V \mid L) & & \downarrow D_{K\cup L} & & H^{p+1}(U \cap V \mid K \cap L) & \longrightarrow \cdots \\
& \downarrow D_{K\cap L} & & \downarrow D_K \oplus (-D_L) & & & & \downarrow D_{K\cap L} & \\
\cdots \longrightarrow & H_{n-p}(U \cap V) & \xrightarrow{\psi_{n-p}} & H_{n-p}(U) \oplus H_{n-p}(V) & \xrightarrow{\varphi_{n-p}} & H_{n-p}(M) & \xrightarrow{\Delta_p} & H_{n-p-1}(U \cap V) & \longrightarrow \cdots
\end{array}$$

and rewrite it, up to isomorphisms, as follows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^p(U\cap V \mid K\cap L) & \overset{\psi_{K\cap L}}{\longrightarrow} & H^p(U\mid K)\oplus H^p(V\mid L) & \overset{\varphi_{K,L}}{\longrightarrow} & H^p(M\mid K\cup L) & \longrightarrow \cdots \\
& \downarrow {\scriptstyle D_{K\cap L}} & & \downarrow {\scriptstyle D_K \oplus (-D_L)} & & \downarrow {\scriptstyle D_{K\cap L}} & \\
\cdots \longrightarrow & H_{n-p}(U\cap V) & \overset{\psi_{n-p}}{\longrightarrow} & H_{n-p}(U)\oplus H_{n-p}(V) & \overset{\phi_{n-p}}{\longrightarrow} & H_{n-p}(M) & \longrightarrow \cdots
\end{array}$$

where the horizontal lines are sequences of Mayer Vietoris, therefore exact.

We have that each compact set in $U\cap V$ is contained in the intersection $K\cap L$ of the compact set $K\subset U$ and $L\subset V$. Similarly, each compact set $U\cup V$ is contained in the union $K\cup L$ of the compact set $K\subset U$ and $L\subset V$. Therefore, we can consider the direct systems of abelian groups and homomorphisms of direct systems:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \{H^p(U\cap V \mid K\cap L)\}_{K\cap L\in A} & \overset{\{\psi_{K\cap L}\}}{\longrightarrow} & \{H^p(U\mid K)\oplus H^p(V\mid L)\}_{K,L\in A} & \overset{\{\varphi_{K,L}\}}{\longrightarrow} & \{H^p(M\mid K\cup L)\}_{K\cup L\in A} & \longrightarrow \cdots \\
& \downarrow {\scriptstyle \{D_{K\cap L}\}} & & \downarrow {\scriptstyle \{D_K \oplus (-D_L)\}} & & \downarrow {\scriptstyle \{D_{K\cap L}\}} & \\
\cdots \longrightarrow & \{H_{n-p}(U\cap V)\} & \overset{\{\psi_{n-p}\}}{\longrightarrow} & \{H_{n-p}(U)\oplus H_{n-p}(V)\} & \overset{\{\phi_{n-p}\}}{\longrightarrow} & \{H_{n-p}(M)\} & \longrightarrow \cdots
\end{array}$$

Passing the limit on the compact set $K\subset U$ and $L\subset V$ and using Proposition 2.9, we have that the following diagram is commutative with exact lines:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^p_c(U\cap V) & \overset{\varinjlim_{K\cap L} \psi_{K\cap L}}{\longrightarrow} & H^p_c(U)\oplus H^p_c(V) & \overset{\varinjlim_{K,L} \varphi_{K,L}}{\longrightarrow} & H^p_c(M) & \longrightarrow & H^{p+1}_c(U\cap V) & \longrightarrow \cdots \\
& \downarrow {\scriptstyle D_{U\cap V} = \varinjlim_{K\cap L} D_{K\cap L}} & & \downarrow {\scriptstyle D_U \oplus (-D_V)} & & \downarrow {\scriptstyle D_M = \varinjlim_{K\cup L} D_{K\cup L}} & & \downarrow {\scriptstyle D_{U\cap V}} & \\
\cdots \longrightarrow & H_{n-p}(U\cap V) & \longrightarrow & H_{n-p}(U)\oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(M) & \longrightarrow & H_{n-p-1}(U\cap V) & \longrightarrow \cdots
\end{array}$$

where $D_U \oplus (-D_V) = \varinjlim_{K} D_K \oplus \varinjlim_{L}(-D_L)$.

$\square$

Lemma 3.3. *Let M be a connected n-manifold and R-orientable such that $M = U\cup V$, where U and V are open sets such that*

$$\begin{aligned}
D_U : H^p_c(U) &\longrightarrow H_{n-p}(U)\\
D_V : H^p_c(V) &\longrightarrow H_{n-p}(V)\\
D_{U\cap V} : H^p_c(U\cap V) &\longrightarrow H_{n-p}(U\cap V)
\end{aligned}$$

are isomorphisms. Then $D_M : H^p_c(M) \longrightarrow H_{n-p}(M)$ is an isomorphism.

Proof. Indeed, by Lemma 3.2, there is a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_c^p(U\cap V) & \longrightarrow & H_c^p(U)\oplus H_c^p(V) & \longrightarrow & H_c^p(M) & \longrightarrow & H_c^{p+1}(U\cap V) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U\oplus -D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} & \\
\cdots \longrightarrow & H_{n-p}(U\cap V) & \longrightarrow & H_{n-p}(U)\oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(M) & \longrightarrow & H_{n-p-1}(U\cap V) & \longrightarrow \cdots
\end{array}$$

such that the horizontal lines are exact. Since the vertical arrows D_U, D_V and $D_{U\cap V}$ are isomorphisms, the *Five Lemma* implies that D_M is also an isomorphism. □

Lemma 3.4. *Let M be a connected n-manifold and R-orientable such that M is the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$ such that for each $i \in \mathbb{N}$, $D_{U_i} : H_c^p(U_i) \to H_{n-p}(U_i)$ is an isomorphism. So $D_M : H_c^p(M) \to H_{n-p}(M)$ is an isomorphism.*

Proof. Since $M = \bigcup_{i\in\mathbb{N}} U_i$, given $K \subset M$ compact, there is $i \in \mathbb{N}$ such that $K \subset U_i$. By Proposition 2.12, we have

$$\varinjlim_{i\in\mathbb{N}} H_{n-p}(U_i) \cong H_{n-p}(M) \qquad (9)$$

Now consider, for each fixed $i \in \mathbb{N}$, the sets $A_i = \{K \subset U_i \mid K \text{ is compact}\}$ and $A = \{K \subset M \mid K \text{ is compact}\}$. Note that $A = \bigcup_{i\in\mathbb{N}} A_i$. In fact, given $K \in A$, there is $i \in \mathbb{N}$ such that $K \subset U_i$, so $K \in A_i \subset \bigcup_{i\in\mathbb{N}} A_i$ which implies that $A \subseteq \bigcup_{i\in\mathbb{N}} A_i$. On the other hand, if $K \in \bigcup_{i\in\mathbb{N}} A_i$ then there is $i_0 \in \mathbb{N}$ such that $K \in A_{i_0}$ and then $K \subset U_{i_0} \subset \bigcup_{i\in\mathbb{N}} U_i = M$. Hence, $K \in A$ which implies that $\bigcup_{i\in\mathbb{N}} A_i \subseteq A$. In addition, it is clear that $A_i \subset A_{i+1}$, for all $i \in \mathbb{N}$.

We claim that for every $K \in A_i$, $H^p(M \mid K) \cong H^p(U_i \mid K)$. In fact, since $K \in A_i$ then K is compact. In addition, $K \subset M$ and M is Hausdorff, hence K is a closed set, and consequently, $M \setminus K$ is an open set in M.

Also note that as $K \in A_i$ then $K \subset U_i \subset M$. Now, taking $A = U_i$ and $B = M\setminus K$ we have $M = U_i \cup (M \setminus K) = \text{int}(U_i) \cup \text{int}\,(M \setminus K) = \text{int}A \cup \text{int}B$.

We also have $A\cap B = (M\setminus K)\cap U_i = U_i\setminus K$. By Excision Theorem, $(A, A\cap B) \hookrightarrow (M, B)$ induces an isomorphism $H^p(M, B) \cong H^p(A, A\cap B)$, i.e., $H^p(M \mid K) \cong H^p(U_i \mid K)$. Hence, as $H_c^p(U_i) = \varinjlim_{K\in A_i} H^p(U_i \mid K)$ then

$$H_c^p(U_i) \cong \varinjlim_{K\in A_i} H^p(M \mid K). \qquad (10)$$

We saw that $A_i \subset A_{i+1}$, for all $i \in \mathbb{N}$. Therefore, it is possible to build (naturally) a homomorphism $\phi_i^{i+1} : H_c^p(U_i) \to H_c^p(U_{i+1})$, such that $\{H_c^p(U_i), \phi_i^{i+1}\}_{i\in\mathbb{N}}$ is a direct system of abelian groups. In fact, given $i \in \mathbb{N}$, we have that inclusion $I : U_i \to U_{i+1}$ induces homomorphism $\left(I_i^{i+1}\right)_* : H_{n-p}(U_i) \to H_{n-p}(U_{i+1})$. By hypothesis, there are isomorphisms $D_{U_i} : H_c^p(U_i) \to H_{n-p}(U_i)$ and $D_{U_{i+1}} : H_c^p(U_{i+1}) \to H_{n-p}(U_{i+1})$. Define $\phi_i^{i+1} = D_{U_{i+1}}^{-1} \circ i_i^{i+1} \circ D(U_i)$.

$$\begin{CD}
H_{n-p}(U_i) @>{\left(I_i^{i+1}\right)_*}>> H_{n-p}(U_{i+1}) \\
@A{D_{U_i}}AA @AA{D_{U_{i+1}}}A \\
H_c^p(U_i) @>>{\phi_i^{i+1}}> H_c^p(U_{i+1})
\end{CD}$$

Of course $\phi_{i+1}^{i+2} \circ \phi_i^{i+1} = \phi_i^{i+2}$. Therefore, $\{H_c^p(U_i), \phi_i^{i+1}\}_{i\in\mathbb{N}}$ is in fact a direct system of abelian groups. Therefore, we can consider the direct limit of $\{H_c^p(U_i), \phi_i^{i+1}\}_{i\in\mathbb{N}}$, namely, $\varinjlim_{i\in\mathbb{N}} H_c^p(U_i)$. Now, for (10), we have

$$\begin{aligned}
\varinjlim_{i\in\mathbb{N}} H_c^p(U_i) &\cong \varinjlim_{i\in\mathbb{N}} \left(\varinjlim_{A_i} H^p(M \mid K) \right) \\
&\cong \varinjlim_{A_i} H^p(M \mid K) \\
&= H_c^p(M),
\end{aligned}$$

i.e.,

$$H_c^p(M) \cong \varinjlim_{i\in\mathbb{N}} H_c^p(U_i). \tag{11}$$

Now, from the hypothesis, we have $D_{U_i} : H_c^p(U_i) \to H_{n-p}(U_i)$ is isomorphism, for all $i \in \mathbb{N}$. Consider the application

$$\{D_{U_i}\} : \{H_c^p(U_i), \phi_i^{i+1}\}_{i\in\mathbb{N}} \to \{H_{n-p}(U_i), \left(j_i^{i+1}\right)_*\}_{i\in\mathbb{N}}.$$

It is possible to show that the diagram

$$\begin{CD}
H_c^p(U_i) @>{D_{U_i}}>> H_{n-p}(U_i) \\
@V{\phi_i^{i+1}}VV @VV{\left(j_i^{i+1}\right)_*}V \\
H_c^p(U_{i+1}) @>>{D_{U_{i+1}}}> H_{n-p}(U_{i+1})
\end{CD}$$

is commutative. Therefore, $\{D_{U_i}\}$ is a morphism between direct systems and therefore induces a morphism

$$\varinjlim_{i\in\mathbb{N}} D_{U_i} : \varinjlim_{i\in\mathbb{N}} H^p_c(U_i) \to \varinjlim_{i\in\mathbb{N}} H_{n-p}(U_i),$$

which, by (9) e (11), we can rewrite as

$$\varinjlim_{i\in\mathbb{N}} D_{U_i} : H^p_c(M) \to H_{n-p}(M).$$

By Corollary 2.6, $\varinjlim_{i\in\mathbb{N}} D_{U_i}(\{x\}) = D_{U_i}(\{x\})$, for all $\{x\} \in H^p_c(M)$. Thus, if $x \in H^p(U_i \mid K)$ then $D_{U_i}(\{x\}) = \mu_K \frown x$. Since $H^p(U_i \mid K) \cong H^p(M \mid K)$ then $x \in H^p(M \mid K)$ and consequently, $D_M(\{x\}) = \mu_K \frown x$. Therefore, $\varinjlim_{i\in\mathbb{N}} D_{U_i}(\{x\}) = D_M(\{x\})$, for all $\{x\} \in H^p_c(M)$ which implies that $D_M = \varinjlim_{i\in\mathbb{N}} D_{U_i}$. Hence, once each $D_{U_i} : H^p_c(U_i) \to H_{n-p}(U_i)$ is an isomorphism, Corollary 2.10 implies that $\varinjlim_{i\in\mathbb{N}} D_{U_i}$ is an isomorphism. Consequently,

$$D_M : H^p_c(M) \to H_{n-p}(M)$$

is an isomorphism. □

Proposition 3.5. *The dual map $D_{\mathbb{R}^n} : H^p_c(\mathbb{R}^n) \to H_{n-p}(\mathbb{R}^n)$ is an isomorphism, for all $p \in \mathbb{Z}^+$.*

Proof. Suppose initially that $p \neq n$. Thus $n-p \neq 0$ and $H_{n-p}(\mathbb{R}^n) = 0$. So, Proposition 2.28 implies that $H^p_c(\mathbb{R}^n) = 0$. Therefore, $D_{\mathbb{R}^n} : H^p_c(\mathbb{R}^n) \to H_{n-p}(\mathbb{R}^n)$ is trivially an isomorphism. Let us check the case $n = p$. We have $n - p = 0$ which implies that $H_{n-p}(\mathbb{R}^n) = H_0(\mathbb{R}^n) \cong \mathbb{Z}$. Furthermore, $D_{\mathbb{R}^n} = \varinjlim_{K\in A} D_K$, where

$$\begin{aligned} D_K : H^n(\mathbb{R}^n \mid K) &\to H_0(\mathbb{R}^n) \\ \varphi &\mapsto \mu_K \frown \varphi \end{aligned}$$

for all $K \in A$. By Proposition 2.28, $D_{\mathbb{R}^n} = \varinjlim_{l\in\mathbb{N}} D_{B[0,l]}$, where in

$$\begin{aligned} D_{B[0,l]} : H^n(\mathbb{R}^n \mid B[0,l]) &\to H_0(\mathbb{R}^n) \\ \varphi &\mapsto \mu_{B[0,l]} \frown \varphi. \end{aligned}$$

We claim that $D_{B[0,l]}$ is an isomorphism, for all $l \in \mathbb{N}$. In fact, consider the generator $\mu_{B[0,l]} \in H^n(\mathbb{R}^n \mid B[0,l])$. It is possible to find a generator g_B of $H^n(\mathbb{R}^n \mid B[0,l]) \cong \mathrm{Hom}_{\mathbb{Z}}(H^n(\mathbb{R}^n \mid B[0,l]), \mathbb{Z})$ such that the cocycle that represents g_B take value 1 over the cycle it represents $\mu_{B[0,l]}$. Therefore, the definition of cap product implies

$$D_{B[0,l]}(g_B) = \mu_{B[0,l]} \frown g_B = g_B(\mu_{B[0,l]}) \cdot \mu_{B[0,l]|_{[v_n]}} = \mu_{B[0,l]|_{[v_n]}} = 1.$$

In other words, $D_{B[0,l]}$ takes generator to generator. Since $H_0(\mathbb{R}^n) \cong \mathbb{Z} \cong H^n(\mathbb{R}^n \mid B[0,l])$, then $D_{B[0,l]}$ is an isomorphism.

Finally, by Corollary 2.10 we have $D_{\mathbb{R}^n}$ is an isomorphism. □

Proposition 3.6. *Suppose that M is an open subset of $\mathbb{R}^n$. The dual map*

$$D_M : H_c^p(M) \to H_{n-p}(M)$$

is an isomorphism. Consequently, the result holds if M is a countable (finite or infinite) union of open sets homeomorphic to $\mathbb{R}^n$.

Proof. Let U_i be a limited convex open set[1] of $\mathbb{R}^n$. Once U_i is homeomorphic to $\mathbb{R}^n$, by Proposition 3.5 the dual map $D_{U_i} : H_c^p(U_i) \to H_{n-p}(U_i)$ is an isomorphism. Now, let U_i and U_j be convex open sets with $i \neq j$. Since $U_i \cap U_j$ is also a convex open set, the dual maps D_{U_i}, D_{U_j} e $D_{U_i \cap U_j}$ are isomorphisms and then Lemma 3.3 ensures that $D_{U_i \cup U_j}$ is an isomorphism. Now, if we define $W_k = U_1 \cup U_2 \cup \cdots \cup U_k$, where U_p is a convex open set for all $p = 1, \dots, k$, using induction over k, the result is also true for W_k.

Now consider open disks $E(a, \epsilon)$, where a has rational coordinates and $\epsilon > 0$ is rational. We know that the collection $\{E(a, \epsilon)\}_{a \in \mathbb{R}^n, \epsilon \in \mathbb{Q}_+^*}$ forms a basis for the topology of $\mathbb{R}^n$. Once M is an open set of $\mathbb{R}^n$, we can find a collection of open convex sets $U_1, U_2, \dots$ such that $M = \bigcup_{i=0}^{\infty} U_i$. If $W_k = \bigcup_{i=0}^{k} U_i$ then $W_i \subset W_{i+1}$ and we can rewrite $M = \bigcup_{i=0}^{\infty} W_k$.

Once $D_{W_k} : H_c^p(W_k) \to H_{n-p}(W_k)$ is an isomorphism, for all $k \in \mathbb{N}$, Lemma 3.4 implies the dual map

$$D_M : H_c^p(M) \to H_{n-p}(M)$$

is an isomorphism.

Now, suppose M is a finite or infinite countable union of open sets homeomorphic to $\mathbb{R}^n$. Just use arguments similar to those used in the previous case, replacing the terms "*limited convex open*" by "*open set of* $\mathbb{R}^n$".

□

Finally, we present the proof of Poincaré's Duality:

[1] For example, an open ball.

Proof of Poincaré Duality: The case $M = \mathbb{R}^n$ is a consequence of Proposition 3.5. The Poincaré Duality version for closed manifolds and for non-compact manifolds with an countable base for their respective topologies is a consequence of Proposition 3.6, once in the latter case we can cover M with an countable family of open homeomorphisms to $\mathbb{R}^n$.

Next, we prove the case where M is an arbitrary non-compact manifold. Then we use *Zorn's Lemma*. Define the following collection:

$$I_M := \{U \text{ open set of } M \mid D_U : H_c^p(U) \to H_{n-p}(U) \text{ is isomorphism}\}.$$

Note that by taking an open set $U \subset M$ such that U is homeomorphic to $\mathbb{R}^n$, Proposition 3.5 implies that D_U is an isomorphism, and then $U \in I_M$. Consequently, I_M is non-empty. It is also clear that I_M is partially ordered by inclusion.

Let $T_M \subset I_M$ be a subset of I_M totally ordered. Define $L := \bigcup_{U \in T_M} U$. By Lemma 3.4, $L \in I_M$, furthermore, for all $U \in T_M$, we have $U \subset L$ which implies that L is a upper quota for T_M. By Zorn's Lemma, I_M has a maximal element, let us say $\tilde{M}$. Since $\tilde{M} \in I_M$, we have that $\tilde{M}$ is an open set of M and $D_{\tilde{M}} : H_c^p(\tilde{M}) \to H_{n-p}(\tilde{M})$ is an isomorphism.

If $M \neq \tilde{M}$, we can choose $x \in M \setminus \tilde{M}$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. By Propositions 3.5 and 3.6, the dual maps D_V and $D_{\tilde{M}\cap V}$ are isomorphisms. Consequently, $D_{\tilde{M}\cup V}$ is also an isomorphism by Lemma 3.3.

On the other hand, since $\tilde{M} \cup V$ is an open set of M hence $\tilde{M} \cup V \in I_M$, which contradicts the maximality of $\tilde{M}$. Therefore, $M = \tilde{M}$ e $D_M : H_c^p(M) \to H_{n-p}(M)$ is an isomorphism. □

3.2. Lefschetz duality

We consider a compact n-manifold M with boundary $\partial M = A \cup B$, where A and B are compact $(n-1)$-dimensional manifolds. The following theorem is a version of Poincaré's Duality for this case. Whenever $A = \emptyset$ ou $B = \emptyset$, this version is known as **Lefschetz duality**. More precisely:

Theorem 3.7 (Lefschetz duality)**.** *Let M be an R-orientable compact n-manifold with boundary such that ∂M can be decomposed as the union of two $(n-1)$-manifolds*[2] *A and B, compact and with common border $\partial A = \partial B = A \cap B$. If we consider the*

[2]We consider the possibility of that A, B or $A \cap B$ are empty sets.

fundamental class $[M] \in H_n(M, \partial M, R)$*, then homomorphism*

$$\begin{array}{rcl} D_M : H^p(M, A; R) & \to & H_{n-p}(M, B; R) \\ \varphi & \mapsto & D_M(\varphi), \end{array}$$

given by $D_M(\varphi) = [M] \frown \varphi$ *is an isomorphism, for each* $p \in \mathbb{Z}^+$.

Proof. Note initially that the dual map D_M is well defined. In fact, since M is R-orientable, then $M \setminus \partial M$ is also R-orientable. Thus, Lemma 2.23 guarantees the existence of a fundamental class relative to $[M]$ in $H_{n-p}(M, \partial M, R)$ which provides guidance at each point of $M \setminus \partial M$. Now, just consider the more general form *cap product* and use the collar neighborhoods of $\partial A, \partial B$ and ∂M. For more details, one can see [16].

We prove the result is true for the case $B = \emptyset$. We have $\partial M = A \cup B = A$ and $\partial A = \partial B = \emptyset$, that is ∂M has no boundary and $H^p(M, A, R) = H^p(M, \partial M, R)$. Once M is compact with boundary, then ∂M has a collar neighborhood in M and, consequently, $H^p(M, \partial M, R) \cong H^p_c(M \setminus \partial M, R)$. Now, we use Theorem 3.1, to guarantee the isomorphism $H^p_c(M \setminus \partial M, R) \cong H_{n-p}(M \setminus \partial M, R)$. Thus, we conclude that $H^p(M, A, R) \cong H_{n-p}(M \setminus \partial M, R)$. Once $H_{n-p}(M \setminus \partial M, R) \cong H_{n-p}(M, R) = H_{n-p}(M, \emptyset, R) = H_{n-p}(M, B, R)$, the result is true for $B = \emptyset$.

The general case reduces to the case where $B = \emptyset$, by applying the Five Lemma to the following diagram (to simplify the notation we omit the ring R):

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^p(M, \partial M) & \longrightarrow & H^p(M, A) & \longrightarrow & H^p(\partial M, A) & \longrightarrow & H^{p+1}(M, \partial M) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow D_M & & \big\downarrow \cong & & \big\downarrow & & \\
 & & & & & & H^p(B, \partial B) & & & & \\
 & & & & & & \big\downarrow D_B & & & & \\
\cdots & \longrightarrow & H_{n-p}(M) & \longrightarrow & H_{n-p}(M, B) & \longrightarrow & H_{n-p-1}(B) & \longrightarrow & H_{n-p-1}(M) & \longrightarrow & \cdots
\end{array}$$

where the top line is the exact sequence in cohomology of the triple $(M, \partial M, A)$ and the bottom line is the exact sequence of the pair (M, B) for the homology.

We write $\partial M = (A \cup B) \cup C$, where $C = \emptyset$. We have $\partial(A \cup B) = \partial(\partial M) = \emptyset = \partial C$. Since $(A \cup B) \cap C = \emptyset$ then $\partial(A \cup B) = \partial C = (A \cup B) \cap C$. For the previous case, the result is true for $C = \emptyset$, thus the dual map

$$\begin{array}{rcl} H^p(M, A \cup B, R) & \to & H_{n-p}(M, R) \\ \varphi & \mapsto & [M] \frown \varphi \end{array}$$

is an isomorphism, i.e., $H^p(M, \partial M, R) \cong H_{n-p}(M, R)$. Likewise, $D_B : H^p(B, \partial B, R) \to H_{n-p-1}(B, R)$ is an isomorphism. On the other hand, we know that $\partial M = A \cup B$. Using collar neighborhood, we can assume that A and B are open sets in ∂M and consider the inclusion $(B, B \cap A) \to (\partial M, A)$. By Excision Theorem, $H^p(B, B \cap A, R) \cong H^p(\partial M, A, R)$. Since $B \cap A = \partial B$, one has $H^p(\partial M, A, R) \cong H^p(B, \partial B, R)$. Therefore, the composition

$$H^p(\partial M, A) \xrightarrow{\cong} H^p(B, \partial B) \longrightarrow H_{n-p-1}(B)$$

is an isomorphism. Once the diagram commutes, Five Lemma implies the dual map D_M is an isomorphism. □

3.3. Alexander duality

Fifteen years after the proof of the first version of Poincaré Duality, [4], Alexander guarantees in [17, 18] that the homology was independent of triangulation. Consequently, it holds for combinatorial manifolds. In [5], Alexander gave a new proof and generalized the Jordan-Brouwer Separation Theorem, nowadays known as *Alexander Duality*. Alexander ensured the existence of an isomorphism between a homology group of the complement of a compact set K in a sphere S^n and a cohomology group of K. More precisely:

Theorem 3.8 (Alexander Duality). *If $K \subset S^n$ is a proper subspace of S^n, such that K is non-empty, compact and locally contractible, then $H_p(S^n \setminus K; \mathbb{Z}) \cong H^{n-p-1}(K, \mathbb{Z})$, for all $p \in \mathbb{Z}^+$.*

Proof. We present a proof just for the case where $p \neq 0$. The case $p = 0$ is simpler and is left to the reader (an idea of the proof can be found for example in [15]). So, after supposing $p \neq 0$, we hide the ring of coefficients $\mathbb{Z}$, for simplicity. Let $A := \{U \text{ open set of } S^n \mid K \subset U\}$. For Poincaré Duality, we have

$$H_p(S^n \setminus K) \cong H_c^{n-p}(S^n \setminus K). \tag{12}$$

Furthermore, by defining $B := \{L \subset S^n \setminus K \mid L \text{ is compact}\}$, we have

$$H_c^{n-p}(S^n \setminus K) = \varinjlim_{L \in B} H^{n-p}((S^n \setminus K) \mid L). \tag{13}$$

Write $K^c := S^n \setminus K$ and define $\tilde{A} := \{L \subset K^c : L \text{ is compact}\}$. Note initially that there is a bijection between the sets $\tilde{A}$ and A. In fact, given $L \in \tilde{A}$, we have $L \subset K^c$ which implies that $K = (K^c)^c \subset L$. Since L is compact then L^c is an open set in S^n, therefore

$L^c \in A$. On the other hand, given $U \in A$ then $K \subset U$ and hence $U^c \subset K^c$. Now, U^c is a closed set in S^n which implies that U^c is compact and thus $U^c \in \tilde{A}$. Therefore, the association $U \mapsto U^c$ is a bijection. Consequently, we can write

$$\varinjlim_{L\in\tilde{A}} H^{n-p}((S^n \setminus K) \mid L) = \varinjlim_{U\in A} H^{n-p}((S^n \setminus K) \mid U^c) = \varinjlim_{U\in A} H^{n-p}(S^n \setminus K, U \setminus K).$$

Since K is a closed set, U is an open set and $K \subset U$, by Excision theorem we have $H^{n-p}(S^n \setminus K, U \setminus K) \cong H^{n-p}(S^n, U)$, for all $U \in A$. Thus,

$$\varinjlim_{A} H^{n-p}(S^n \setminus K, U \setminus K) \cong \varinjlim_{A} H^{n-p}(S^n, U). \tag{14}$$

Given $U \in A$, consider the exact long sequence of the pair (S^n, U):

$$\cdots \longrightarrow H^{n-p-1}(S^n) \longrightarrow H^{n-p-1}(U) \longrightarrow H^{n-p}(S^n, U) \longrightarrow H^{n-p}(S^n) \longrightarrow \cdots$$

Since $p \neq 0$ then $H^{n-p}(S^n) = 0$, therefore $H^{n-p-1}(U) \cong H^{n-p}(S^n, U)$, for all $U \in A$. Consequently, we have

$$\varinjlim_{A} H^{n-p}(S^n, U) = \varinjlim_{A} H^{n-p-1}(U). \tag{15}$$

We show that $\varinjlim_{A} H^i(U) = \tilde{H}^i(K)$, for all $i \in \mathbb{Z}^+$. Let K be a subset of compacting S^n of $\mathbb{R}^n$ and any $i \in \mathbb{Z}^+$. Since K is locally contractible then it is also a retract of some neighborhood[3] U_0 in S^n. To compute the direct limit, we can restrict attention to the open set $U \in A$ such that $U \subset U_0$. Consequently,

$$\varinjlim_{A} H^i(U) \cong \varinjlim_{U\in A, U\subset U_0} H^i(U) = \varinjlim_{A'} H'(U),$$

where $A' = \{U \in A : U \subset U_0\}$. Note that, given $U \in A'$, we have that K is a retract of U. In fact, since K is a retract of U_0, there is a retraction $\gamma_{U_0} : U_0 \to K$. Since $U \subset U_0$ and $K \subset U$, we can consider the restriction $\gamma_U = \gamma_{U_0}|_U : U \to K$. Therefore γ_U is a retraction. With this, we can show that the restriction morphism

$$\Theta : \varinjlim_{A'} H'(U) \to H'(K)$$

[3]A compact subspace $K \subset \mathbb{R}^n$ is a retract of some neighborhood in $\mathbb{R}^n$ if and only if K is locally contractible, see [15, Teorema A7].

is surjective, injective, and therefore an isomorphism. Therefore,

$$\lim_{\overrightarrow{A}} H^i(U) = \lim_{\overrightarrow{A'}} H^i(U) \cong H^i(K). \tag{16}$$

Finally, the sequence of isomorphisms

$$\begin{aligned} H_p(S^n \setminus K) &\overset{(12)}{\cong} H_c^{n-p}(S^n \setminus K) \\ &\overset{(13)}{\cong} \lim_{\overrightarrow{A}} H^{n-p}(S^n \setminus K, U \setminus K) \\ &\overset{(14)}{\cong} \lim_{\overrightarrow{A}} H^{n-p}(S^n, U) \\ &\overset{(15)}{\cong} \lim_{\overrightarrow{A}} H^{n-p-1}(U) \\ &\overset{(16)}{\cong} H^{n-p-1}(K), \end{aligned}$$

shows that the result is true for $p \neq 0$, as we wanted to show. □

The next type of duality is useful in the topological classification of the link of the singularity at the origin. See Lemma 4.7 for more details.

Theorem 3.9. *Let M be a orientable n-manifold and $K \subset M$ a compact, locally contractible subspace. There are isomorphisms $H_p(M \mid K) \cong H^{n-p}(K)$, for all $p \in \mathbb{Z}$.*

Proof. Let U be an open neighborhood of K in M and V the complement of a compact set in M. We assume $U \cap V = \emptyset$. By Excision Theorem, we have

$$H_p(M \mid K) \cong H_p(U \mid K). \tag{17}$$

If we consider $A = U$, $B = V$, we have that $A \cup B$ is the union of open sets. Consequently, $(A \cup B, A, B)$ is an excisive triad. In addition, $A \cap B = U \cap V = \emptyset$ and again by Excision Theorem, the inclusion of pairs $(A, A \cap B) \hookrightarrow (A \cup B, B)$ induces a isomorphism $H^{n-p}(A \cup B, B) \cong H^{n-p}(A, A \cap B)$. Therefore,

$$H^{n-p}(U \cup V, V) \cong H^{n-p}(U, \emptyset) \cong H^{n-p}(U). \tag{18}$$

Now consider the exact homology sequence of the pair $(M, M \setminus K)$:

$$\cdots \longrightarrow H_p(M \setminus K) \longrightarrow H_p(M) \longrightarrow H_p(M \mid K) \cong H_p(U \mid K) \longrightarrow \cdots$$

Also consider the triple $(M, U \cup V, V)$ and the inclusions

$$(U \cup V, V) \xrightarrow{I} (M, V) \xrightarrow{J} (M, U \cup V).$$

These inclusions induce an exact sequence in cohomology

$$H^{n-p}(M, U \cup V) \longrightarrow H^{n-p}(M, V) \longrightarrow H^{n-p}(U \cup V, V) \cong H^{n-p}(U).$$

Now, considering the isomorphisms in (17, 18) and *cap product* with the fundamental class of M, we obtain the commutative diagram below:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{n-p}(M, U \cup V) & \longrightarrow & H^{n-p}(M, V) & \longrightarrow & H^{n-p}(U) & \longrightarrow \cdots \\ & \downarrow {\scriptstyle M\frown} & & \downarrow {\scriptstyle M\frown} & & \downarrow & \\ \cdots \longrightarrow & H_p(M \setminus K) & \longrightarrow & H_p(M) & \longrightarrow & H_n(U \mid K) & \longrightarrow \cdots \end{array}$$

By Poincaré Duality, taking the direct limit on $U \supset K$ and V, the first two vertical arrows become isomorphisms

$$H_c^{n-p}(M \setminus K) \xrightarrow[\cong]{} H_n(M \setminus K)$$

and

$$H_c^{n-p}(M) \xrightarrow[\cong]{} H_p(M).$$

Considering the commutative diagram above and the vertical isomorphisms, the Five Lemma guarantees that for U open set in M, we have an isomorphism

$$H_p(M \mid K) \cong \varinjlim_{U \supset K} H^{n-p}(U). \tag{19}$$

Note that K is a retract of a neighborhood U_0 of M. In fact, in order to obtain a retraction, we initially built a map $M \hookrightarrow \mathbb{R}^s$ that is a topological embedding in the neighborhood of the compact set K, with s sufficiently large. So, without loss of generality, we can consider K as a compact of $\mathbb{R}^n$, locally contractible. Consequently, K is a retract of some neighborhood of $\mathbb{R}^n$.

Now, we follow the same steps used in the proof of Theorem 3.8, to obtain the isomorphism $\varinjlim_{U \supset K} H^{n-p}(U) \cong H^{n-p}(K)$, and therefore the desired result follows from (19). $\square$

4. The exotic n-spheres

In [10], Milnor presented the first seven examples of 7-dimensional exotic spheres, which give rises to questions related to the existence of extra exotic structures over 7-sphere, and also about exotic structures on spheres in other dimensions. Several techniques were developed and published, for instance [13, 19, 20, 14, 21, 22, 23, 16].

In [13], for the case $n \geq 5$, Kervaire and Milnor studied the group of h cobordism classes of oriented homotopy of n-spheres Θ_n, which is finite and abelian. For $n \neq 4$, it is well known that Θ_n is isomorphic to the group of equivalence classes of smooth structures on n-spheres, which is the classes of oriented smooth n-manifolds which are homeomorphic to the n-sphere, taken up to orientation-preserving diffeomorphism, and the operation is the connected sum (for more details, see [24]). Consequently, the properties of Θ_n provide a remarkable knlowledge about the existence of n-dimensional exotic spheres. For instance, concerning to the order of Θ_n, we already know that $|\Theta_k| = 1$, for $k \in \{1, 2, 3, 5, 6, 12, 61\}$, i.e, each of S^1, S^2, S^3, S^5, S^6, S^{12} and S^{61} only has a unique smooth structure. On the other hand, $|\Theta_{15}| = 16256$, i.e., S^{15} admits 16255 exotic structures. For more details, see [19, 20].

Also in [13], Kervaire and Milnor ensure the existence of the cyclic subgroup of classes of homotopy n-spheres $bP_{n+1} \subset \Theta_n$ that bound parallelizable manifolds (which are manifolds with trivial tangent bundle). It is trivial if n is even. If $n \equiv 1 \mod 4$, it has order 1 or 2. More precisely, in [25] Browder proved that it has order 2 if $n \equiv 1 \mod 4$ is not of the form $2^k - 3$.

Motivated by work [21], Egbert Brieskorn introduced in [22] and [23] the *Brieskorn-Pham manifolds*, which are example of exotic spheres. Brieskorn considered a complex analytic variety V of complex dimension n in some affine space $\mathbb{C}^N$, with a unique singular point at P, defined by one single equation, i.e., a *singular complex hypersurface* V. Its link $K := V \cap S^{2N-1}_{\epsilon}$, which is independent of $\epsilon > 0$ sufficiently small (up to isotopies), is a Brieskorn-Pham manifold, as the next theorem shows (here we use the same statement as in [26]):

Theorem 4.1. *[23] Every exotic sphere of dimension $m = 2n - 1 > 6$ that bounds a paralelizable manifold is the link K of some singular complex hypersurface of the form $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} = 0$, for some appropriate integers $a_j \geq 2$, $j = 1, \cdots, n$.*

For instance, all the 28 possible smooth structures on the oriented 7-sphere (including the standard euclidean sphere), which bound a parallelizable manifold, are given by the links of the singular complex hypersurface of the form $z_1^2 + z_2^2 + z_3^2 + z_4^3 + + z_5^{6p-1} = 0$, where $p = 1, 2, \cdots, 28$.

4.1. The link as a homotopy sphere

Let $f : U \subset \mathbb{C}^{n+1} \to \mathbb{C}$ be a representative of an analytic complex germ with U an open set in $\mathbb{C}^{n+1}$, $f(0) = 0$ and $V_f = f^{-1}(0)$. Consider the set $K_\varepsilon = V_f \cap S_\varepsilon^{2n+1}$, which is called the *link* of the singularity at the origin. In [14], Milnor improved techniques developed by Pham, Brauner, Brieskorn, and Hirzebruch to find conditions under which the link K_ε is a representative of an equivalent class of smooth structure on n-spheres which bound a parallelizable manifold. Namely, an element of the cyclical subgroup $bP_{n+1} \subset \Theta_n$ formed by the homotopy classes of n-spheres that bound parallelizable manifolds. More precisely, Milnor proved the following results.

Proposition 4.2. *[14, Corollary 2.9] There exists small enough $\varepsilon_0 > 0$ so that every sphere $S_\varepsilon^{2n+1} \subset \mathbb{C}^{n+1}$ centered at $0 \in \mathbb{C}^{n+1}$ with $0 < \varepsilon \leq \varepsilon_0$, intersects $V_f \setminus \{0\}$ transversally. Moreover, there is a smooth 1-parameter family of diffeomorphisms $\{\gamma_t\}$, $t \in [0, \varepsilon)$, such that γ_0 is the idently and if $S_{\varepsilon-t}^{2n+1}$ denotes the sphere of radius $\varepsilon - t$, then each γ_t carries the pair $(S_\varepsilon^{2n+1}, K_\varepsilon)$ into $(S_{\varepsilon-t}^{2n+1}, K_{\varepsilon-t})$.*

As we just have seen, once the link of the singularity at the origin K_ε is independent of the radius ε (up to diffeomorphisms), we start to denote it only by K.

Theorem 4.3 (Milnor fibration - Complex case)**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic map germ. There exists a small enough real number $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$,*

$$\phi := \frac{f}{\|f\|} : S_\varepsilon^{2n+1} \setminus K \to S^1 \tag{20}$$

is a smooth projection of a locally trivial fiber bundle, which is independent of the choices of small enough $\varepsilon > 0$, up to diffeomorphisms. Each fiber $F_\theta = \phi^{-1}\left(e^{i\theta}\right)$, where $e^{i\theta} \in S^1$, is a smooth parallelizable manifold $(2n)$-dimensional, with the homotopy type of a n-dimensional CW-complex.

Besides, whenever the origin is an isolated critical point of f, Milnor associated for it a multiplicity denoted by $\mu(f)$, later named by several authors *the Milnor number of the singularity*, given by the topological degree of the map

$$\epsilon \frac{\nabla f}{\|\nabla f\|} : S_\epsilon^{2n+1} \to S_\epsilon^{2n+1},$$

and proved the following result.

Theorem 4.4. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic map germ with isolated critical point at the origin. Then each fiber F_θ has the same homotopy type of a bouquet of n-dimensional spheres $\bigvee_{i=1}^{\mu(f)} S_i^n$, with $\mu(f)$ spheres on the bouquet. Each fiber can be considered as the interior of a smooth compact manifold with boundary, $\bar{F}_\theta = F_\theta \cup K$, where the common boundary K is an $(n-2)$-connected compact manifold $(2n-1)$-dimensional.*

In other words, all the fibers F_θ, fit around their common boundary K and the smooth manifold K is connected if $n = 2$, and simply connected if $n \geq 3$.

Therefore, the fact the fibers F_θ be parallelizable manifolds together with Theorem 4.4 give that the link K represents an element of the subgroup bP_{2n}, provided K is a homotopy sphere.

Now we present relations between the Poincaré Duality and the study of the existence of exotic structures on spheres. We recall some conditions (such as sufficiency, necessity, and computability) found by Milnor so that K be homeomorphic to the $(2n-1)$-sphere and, consequently, a homotopy sphere (see [14, Chapter 8]).

In the following, we assume that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a complex analytic map germ with isolated critical point at the origin and $n \geq 1$. Recall that a *homology sphere* is a d-dimensional manifold having the homology groups of a d-sphere.

Lemma 4.5. *For $n \neq 2$, the link K is homeomorphic to the $(2n-1)$-sphere if and only if it is a homology sphere.*

Proof. First we consider the case $n \geq 3$. If K has the homology of the sphere, one has that

$$\tilde{H}_j(K, \mathbb{Z}) \cong \tilde{H}_j(S^{2n-1}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } j = 2n-1 \\ 0, & \text{if } j \neq 2n-1 \end{cases} .$$

Since Moore spaces are unique[4], (up to homotopy), one concludes that K and S^{2n-1} are homotopy equivalent. Thus, Theorem 4.4 implies that the compact manifold K is simply connected with dimension $2n-1 \geq 5$. Therefore, by the generalized Poincaré conjecture, K is homeomorphic to a sphere S^{2n-1}.

Next, assume that $n = 1$. One has $\tilde{H}_1(K) \cong \mathbb{Z}$ and $\tilde{H}_j(K) \ncong \mathbb{Z}$, for all $j \neq 1$. Therefore, it follows from the classification of low dimension manifolds that K is homeomorphic to S^1.

The converse is trivial. □

Remark 4.6. As explained by Milnor in [14], if we consider $n = 2$ and the Brieskorn polynomial $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$, Hirzebruch points out that the corresponding link K is a homology 3-sphere, but $|\pi_1(K)| = 120$. Therefore, K cannot be homeomorphic to S^3. Consequently, for $n = 2$ the corresponding statement to Lemma 4.5 is false.

Lemma 4.7. *For $n \neq 2$ the link K is homeomorphic to the $(2n-1)$-sphere if and only if the reduced homology group $\tilde{H}_{n-1}(K)$ is trivial.*

[4]See [15, p.368]

Proof. As we have seen, the link K is a orientable and compact manifold $(2n-1)$-dimensional. So, the Poincaré's Duality implies $H^q(K) \cong H_{2n-1-q}(K)$ and Theorem 3.9 ensures that $H_{(2n+1)-q}(S_\varepsilon \mid K) \cong H^q(K)$. Hence, after rewriting $i+2 = 2n-q+1$, one has

$$H_i(K) \cong H_{i+2}(S_\varepsilon \mid K),$$

for all i.

Consider the exact sequence of the pair $(S_\varepsilon, S_\varepsilon \setminus K)$:

$$\cdots \longrightarrow H_t(S_\varepsilon \setminus K) \longrightarrow H_t(S_\varepsilon) \longrightarrow H_t(S_\varepsilon \mid K) \longrightarrow H_{t-1}(S_\varepsilon \setminus K) \longrightarrow \cdots$$

One has that $S_\varepsilon \setminus K$ is a CW-complex such that $\dim(S_\varepsilon \setminus K) \leq n+1$. Therefore, $H_t(S_\varepsilon \setminus K) = 0$, for all $t > n+1$ and $H_t(S_\varepsilon) \cong H_t(S_\varepsilon, S_\varepsilon \setminus K)$ for all $t > n+1$. Consequently,

$$H_{t-2}(K) \cong H_t(S_\varepsilon),$$

for all $t \geq n+2$ and one has:

$$H_j(K) \cong \begin{cases} 0, \text{ if } n \leq j \leq 2n-2 \quad (n+2 \leq t \leq 2n) \\ \mathbb{Z}, \text{ if } j = 2n-1 \quad (t = 2n+1) \\ 0, \text{ if } j > 2n-1 \quad (t > 2n+1) \end{cases}.$$

Now, Theorem 4.4 implies that K is connected if $n = 2$, and simply connected if $n \geq 3$, which implies that $H_0(K) \cong \mathbb{Z}$.

By Hurewicz's Isomorphism Theorem, $H_j(K) = 0$, for all $j < n-1$ and by hypothesis, $H_{n-1}(K) = 0$. Therefore,

$$H_j(K) \cong \begin{cases} \mathbb{Z}, \text{ if } j = 0 \text{ e } j = 2n-1 \\ 0, \text{ if } j \in \mathbb{Z} \setminus \{0, 2n-1\} \end{cases}.$$

Where one does conclude that K is a homology sphere. Now, the result follows from Lemma 4.5. □

Next, choose an orientation for the $2n$-dimensional orientable manifold F_θ and consider, for any two n-dimensional homology class $\alpha, \beta \in H_n(F_\theta)$, the *intersection* *number* $s(\alpha,\beta)$. Now, one can to consider the *intersection pairing*

$$\begin{array}{rcl} s: H_n(F_\theta)\oplus H_n(F_\theta) & \to & \mathbb{Z} \\ \alpha\oplus\beta & \mapsto & s(\alpha,\beta). \end{array}$$

For more detail, see [27, p.59].

Lemma 4.8. *For* $n \neq 2$ *the link* K *is homeomorphic to the* $(2n-1)$*-sphere if and only if the intersection pairing*

$$\begin{array}{rcl} s: H_n(F_\theta)\oplus H_n(F_\theta) & \to & \mathbb{Z} \\ \alpha\oplus\beta & \mapsto & s(\alpha,\beta), \end{array}$$

has determinant ± 1.

Proof. Consider the exact sequence in homology of the pair $(\bar{F}_\theta, K)$:

$$H_n(\bar{F}_\theta) \xrightarrow{j_*} H_n(\bar{F}_\theta, K) \xrightarrow{\partial} H_{n-1}(K) \longrightarrow H_{n-1}(\bar{F}_\theta)$$

Since F_θ is $(n-1)$-connected, the Hurewicz Theorem implies that $H_q(F_\theta)=0$, for all $q \le n-1$. Consequently, $0 = H_{n-1}(F_\theta) \cong H_{n-1}(\bar{F}_\theta)$. Hence, the exact sequence in homology of the pair $(\bar{F}_\theta, K)$, becomes:

$$H_n(\bar{F}_\theta) \xrightarrow{j_*} H_n(\bar{F}_\theta, K) \xrightarrow{\partial} H_{n-1}(K) \longrightarrow 0$$

Now, the Poincaré Duality ensure that

$$H_{2n-q}(\bar{F}_\theta, K) \cong H^q(\bar{F}_\theta),$$

for all integer q. After choose $q = n$, one has

$$H_n(\bar{F}_\theta, K) \cong H^n(\bar{F}_\theta). \tag{21}$$

Since $H_n(\bar{F}_\theta) \cong H_n(F_\theta) \cong \bigoplus_{i=0}^{\mu}\mathbb{Z}$ and $H_{n-1}(\bar{F}_\theta) = 0$, then Universal Coefficient Theorem for Homology guarantees that

$$H^n(\bar{F}_\theta) \cong \mathrm{Ext}(H_{n-1}(\bar{F}_\theta),\mathbb{Z}) \oplus \mathrm{Hom}(H_n(\bar{F}_\theta),\mathbb{Z}) \cong 0 \oplus \frac{H_n(\bar{F}_\theta)}{\mathrm{Torsion}\left(H_n(\bar{F}_\theta)\right)} \cong H_n(\bar{F}_\theta).$$

Therefore, the above sequence of isomorphisms and (21) imply $H_n(\bar{F}_\theta, K) \cong H_n(\bar{F}_\theta)$. Consequently, one concludes that $H_n(\bar{F}_\theta, K)$ is torsion free, has rank μ and:

(i) the intersection pairing

$$\begin{aligned} s' : H_n(\bar{F}_\theta, K) \oplus H_n(\bar{F}_\theta) &\rightarrow \mathbb{Z} \\ \alpha \oplus \beta &\mapsto s'(\alpha, \beta), \end{aligned}$$

has determinant ± 1,

(ii) $s(\alpha, \beta) = s'(j_*(\alpha), \beta)$.

Let us assume that K is homeomorphic to the $(2n-1)$-sphere. In this case, K is a homological sphere and the equality $H_{n-1}(K) = H_n(K) = 0$, implies that $j_* : H_n(\bar{F}_\theta) \rightarrow H_n(\bar{F}_\theta, K)$ is an isomorphism. Since $H_n(F_\theta) \cong H_n(\bar{F}_\theta)$, after identifying $\alpha \equiv j_*(\alpha)$, one has $s(\alpha, \beta) = s'(\alpha, \beta)$. Consequently, condition (i) implies that determinant of s is ± 1.

Conversely, assume that the determinant of s is ± 1. Hence, j_* is an isomorphism and using the exact sequence in the homology of the pair $(\bar{F}_\theta, K)$, we conclude that the manifold K is a homological sphere. Now, the result follows from Lemma 4.5.

□

The last result is a computational criterion that helps us to decide when the link is homeomorphic to the sphere. Before, let us consider the following definition.

Definition 4.9. Consider the monodromy of the locally trivial fiber bundle (20) and the induced representation in the middle homology of the fiber $F_0 = \phi^{-1}(1)$:

$$h_* : H_n(F_0) \rightarrow H_n(F_0).$$

The *characteristic polynomial* of the monodromy is given by $\Delta(t) = \det(tI_* - h_*)$. It can also be called characteristic polynomial of the typical fiber.

Remark 4.10. Note that $\Delta(t)$ is a polynomial of the form $t^\mu + a_1 t^{\mu-1} + \cdots + a_{\mu-1} t \pm 1$, with integer coefficients.

Theorem 4.11. *For $n \neq 2$ the link K is homeomorphic to the $(2n-1)$-sphere if, and only if*

$$\Delta(1) = \pm 1.$$

Proof. Wang's Lemma implies the exactness of the following sequence

$$\cdots \longrightarrow H_{n+1}(S_\varepsilon \setminus K) \longrightarrow H_n(F_0) \xrightarrow{h_* - I_*} H_n(F_0) \longrightarrow H_n(S_\varepsilon \setminus K) \longrightarrow \cdots$$

The Alexander's Duality ensure that

$$H_{n+1}(S_\varepsilon \setminus K) \cong H^{(2n-1)-(n+1)-1}(K) = H^{n-1}(K).$$

On the other hand, by Poincaré Duality one has the following isomorphism

$$H^{n-1}(K) \cong H_{(2n-1)-(n-1)}(K) = H_n(K).$$

Consequently, $H_{n+1}(S_\varepsilon \setminus K) \cong H_n(K) = 0$ and one can rewrite the exact sequence

$$0 \longrightarrow H_n(F_0) \xrightarrow{h_*-I_*} H_n(F_0) \longrightarrow H_{n-1}(K).$$

Thus, Lemma 4.7 implies that the previous exact sequence becomes

$$0 \longrightarrow H_n(F_0) \xrightarrow{h_*-I_*} H_n(F_0) \longrightarrow 0,$$

(i.e., $h_* - I_*$ is isomorphism and $\det(I_* - h_*)$ is invertible), if and only if K is homeomorphic to the $(2n-1)$-sphere. Since $\det(I_* - h_*) \in \mathbb{Z}$, then it is invertible if and only if $\det(I_* - h_*) = \pm 1$. Therefore, K is homeomorphic to the $(2n-1)$-sphere if and only if, $\Delta(1) = \pm 1$. $\square$

We finish this paper by presenting Kervaire's exotic 9-sphere.

Example 4.12. Consider the Brieskorn-Pham polynomial $f(z_1, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots = z_{n+1}^{a_{n+1}}$, with $a_1 = \cdots = a_n = 2$ and $a_{n+1} = 3$. It is possible to show that $\Delta(t) = t^2 - t + 1$, for n odd. Consequently, one has $\Delta(1) = 1$ and Theorem 4.11 ensures that the manifold K is homeomorphic to the $(2n-1)$-sphere, with $2n-1 = 1, 5, 9, 13, \ldots$.

Since f has isolated critical point at the origin, the Theorem 4.4 ensure that K represent a element of the subgroup $bP_{2n} \subset \Theta_{2n-1}$. As we have seen, $|\Theta_1| = 1$ and $|\Theta_5| = 1$ that is, the spheres S^1 and S^5 have a single smooth structure, not allowing exotic structures. Then for the cases $n = 1$ and $n = 3$, the manifold K is diffeomorphic to the standard sphere S^1 or S^5. But, for $n = 5$ one can use the Kervaire invariant $c(F_0) \in \mathbb{Z}_2$ to ensure that the 9-dimensional manifold K is diffeomorphic to Kervaire's exotic 9-sphere.

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