On Frenet Apparatus of Curves in $\mathbb{R}^n$

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Abstract. *In this paper, we present new explicit and nonrecursive formulas for the curvatures and the frame of Frenet of a regular curve with an arbitrary parameter in the Euclidean space $\mathbb{R}^n$, $n > 2$, expressed only in terms of its derivatives.*

Keywords – *curve, Frenet apparatus, Frenet equations, curvatures.*

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Introduction

In his book, Elementary Differential Geometry ([10]), Barrett O'Neill, makes the following remark:

> **... However, for explicit numerical computations — and occasionally for the theory as well — this transference is impractical, since it is rarely possible to find explicit formulas for $\overline{\alpha}$. (For example, try to find a unit-speed parametrization for the curve $\alpha(\mathbf{t}) = (\mathbf{t}, \mathbf{t^2}, \mathbf{t^3})$.)**

Such an observation concerns the explicit calculation of the Frenet apparatus of a curve α, without using its arc length parametrization, that is, using an arbitrary parameter, as in the cited example. One of our goals here is to generalize this calculation for curves in $\mathbb{R}^n$.

Next, we use the material contained mainly in [8], [5], [6] and [12]. For this, let $f : I \longrightarrow \mathbb{R}^n$, $n > 1$, a parametrized $(n-1)$-regular[1] curve, that is, the derivatives $f'(t)$, $f''(t)$, ..., $f^{(n-1)}(t)$ are linearly independent, for all t in the interval I. In this case, looking carefully in the above references, we get $(n-1)$ real functions, defined in I, $\kappa_1(t)$, $\kappa_2(t)$, ..., $\kappa_{n-1}(t)$, $\kappa_j > 0$, $j < n-1$, and a positively oriented orthonormal frame

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[1] f is said to be k-regular if $f'(t)$, $f''(t)$, ..., $f^{(k)}(t)$ are linearly independent.

field along f, $\mathcal{F}_t = \{V_1(t), V_2(t), \ldots, V_n(t)\}$, the *Frenet frame of* f. The functions κ_j, $1 \leq j \leq n-1$, will be called of *curvatures* of f. When $n = 3$, the last curvature, κ_2, is called *torsion* and is indicate by τ. The set

$$\mathcal{A}_f(t) = \{\kappa_1(t), \kappa_2(t), \ldots, \kappa_{n-1}(t), V_1(t), V_2(t), \ldots, V_n(t)\}$$

is called *Frenet apparatus* of f. The elements of this set satisfy (1) below, known as *Frenet equations*, in which $\nu(t) = \|f'(t)\|$ denotes the speed of f and $V_1(t) = \frac{f'(t)}{\nu(t)}$ is the unit tangent field. For simplicity, henceforth, we omit the parameter t.

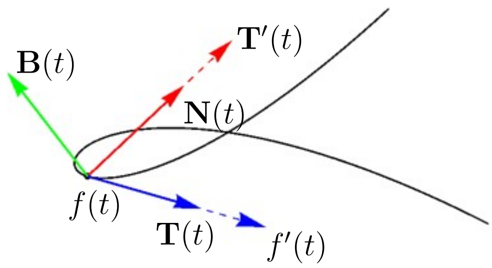
$$\begin{cases} V_1' = \kappa_1 \nu\, V_2 \\ V_j' = -\kappa_{j-1}\nu\, V_{j-1} + \kappa_j \nu\, V_{j+1}, \quad 2 \leq j \leq n-1 \\ V_n' = -\kappa_{n-1}\nu\, V_{n-1}. \end{cases} \tag{1}$$

Furthermore, $\{V_1, \ldots, V_{n-1}\}$ is the Gram-Schmidt orthonormal set constructed from the derivatives f', f'',...,$f^{(n-1)}$ (see the Theorem 1.1) and $V_n = V_1 \times V_2 \times \ldots \times V_{n-1}$. Hence, for each j, $1 \leq j \leq n-1$, and $t \in I$, the space generated by $\{f', f'', \ldots, f^{(j)}\}$ coincides with that generated by $\{V_1, \ldots, V_j\}$. Note that this last fact can be rewritten by using multivector (j-vector) objects, as

$$f' \wedge f'' \wedge \cdots \wedge f^{(j)} = \lambda\, V_1 \wedge \cdots \wedge V_j,$$

where $\lambda(t) \neq 0$, for all $t \in I$. In Theorem 2.1, we make explicit λ in terms of the curvatures of f. When $n = 3$, we use the classic notation: $\kappa_1 = \kappa$, $\kappa_2 = \tau$, $V_1 = \mathbf{T}$, the unitary tangent vector, $V_2 = \mathbf{N}$, the principal normal vector, and $V_3 = \mathbf{B} = \mathbf{T} \times \mathbf{N}$, the binormal vector. Thus, in this case, we have the classic Frenet equations:

$$\begin{cases} \mathbf{T}' = \kappa\, \nu\, \mathbf{N} \\ \mathbf{N}' = -\kappa\, \nu\, \mathbf{T} + \tau\, \nu\, \mathbf{B} \\ \mathbf{B}' = -\tau\, \nu\, \mathbf{N}. \end{cases},$$



or, in the matrix form,

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \nu \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Now, for $n = 5$, we get

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \\ V_5' \end{pmatrix} = \nu \begin{pmatrix} 0 & \kappa_1 & 0 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 & 0 \\ 0 & 0 & -\kappa_3 & 0 & \kappa_4 \\ 0 & 0 & 0 & -\kappa_4 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix}. \tag{2}$$

1. Basic Facts on Wedge Product

Given the p-vectors $v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \in \bigwedge^p(\mathbb{R}^n)$, suppose that

$$(v_1\ v_2\ \ldots\ v_p) = (w_1\ w_2\ \ldots\ w_p)A,$$

for some $p \times p$ matrix $A = (a_{ij})$ (see Fact 3.5). This means that each v_j is a linear combination of the vectors $w_1, w_2, \ldots, w_p$. More precisely,

$$v_j = \sum_{i=1}^{p} a_{ij} w_i.$$

It is easy to see that

$$v_1 \wedge \cdots \wedge v_p = (\det A) w_1 \wedge \cdots \wedge w_p. \tag{3}$$

In particular, if A is a triangular matrix,

$$v_1 \wedge \cdots \wedge v_p = a_{11} a_{22} \ldots a_{pp}\, w_1 \wedge \cdots \wedge w_p. \tag{4}$$

Another well known fact is that

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_p) = \det(v_i \cdot w_j), \tag{5}$$

known as Gram determinant, defines an intern product in $\bigwedge^p(\mathbb{R}^n)$ and, thus, we have the a well defined norm:

$$\|v_1 \wedge \cdots \wedge v_p\| = \sqrt{\det(v_i \cdot v_j)}, \tag{6}$$

which is also known as *area of parallelogram generated by the vectors* $v_1, v_2, \ldots, v_p$. It is convenient to remark that the $(p \times p)$ matrix $(v_i \cdot v_j)$ equals ${}^{\mathrm{T}}(v_1\ v_2\ \ldots\ v_p)(v_1\ v_2\ \ldots\ v_p)$, where ${}^{\mathrm{T}}M$ denotes the transpose of M and $(v_1\ v_2\ \ldots\ v_p)$ is the $(n \times p)$ matrix whose columns are the column vectors $v_1, \ldots, v_p$. Thus, when $p = n$, we get a useful identity, namely

$$\|v_1 \wedge \cdots \wedge v_n\| = |\det(v_1\ v_2\ \ldots\ v_n)|.$$

In fact,

$$\det({}^{\mathrm{T}}(v_1\ v_2\ \dots\ v_n)(v_1\ v_2\ \dots\ v_n)) = \det(v_i \cdot v_j).$$

Now, we consider a $(n-1)$-vector in $\bigwedge^{n-1}(\mathbb{R}^n)$, say $w = w_1 \wedge \cdots \wedge w_{n-1}$. In this case, we also have the *cross product* $\widetilde{w} = w_1 \times \cdots \times w_{n-1}$, which belongs to $\mathbb{R}^n$ and, for all $X \in \mathbb{R}^n$,

$$\widetilde{w} \cdot X = \det(w_1, \dots, w_{n-1}, X)$$

holds. This implies the

$$\{w_1, \dots, w_{n-1}, w_1 \times \cdots \times w_{n-1}\}$$

is a positively oriented basis, whenever $\{w_1, \dots, w_{n-1}\}$ is linearly independent. In fact, $\widetilde{w} \neq 0$ and

$$0 < \widetilde{w} \cdot \widetilde{w} = \det(w_1, \dots, w_{n-1}, w_1 \times \cdots \times w_{n-1}).$$

Of course we cannot compare w with $\widetilde{w}$, however two $(n-1)$-vectors are equal if, and only if, the corresponding cross products are equal. This comes from the fact that the coordinates of $v_1 \wedge \cdots \wedge v_{n-1}$, in the canonical basis of $\bigwedge^{n-1}(\mathbb{R}^n)$, coincide, up to signal, with those of the $v_1 \times \cdots \times v_{n-1}$. Indeed,

$$v_1 \times \cdots \times v_{n-1} = *(v_1 \wedge \cdots \wedge v_{n-1}),$$

where $*$ is the Hodge star operator.

Now, for each $m \in \mathbb{Z}$, $2 \leq m \leq n$, we introduce a very useful operator, denoted by Φ_m, which we will call m-*Gram-Schmidt operator*. This operator is a bilinear operator that acts on the product $\bigwedge^{m-1}(\mathbb{R}^n) \times \bigwedge^m(\mathbb{R}^n)$. Initially, we define $\phi : (\mathbb{R}^n)^{m-1} \times (\mathbb{R}^n)^m \longrightarrow \mathbb{R}^n$ by the vector determinant

$$\phi(v_1, \dots, v_{m-1}, w_1, \dots, w_m) = \det \begin{pmatrix} v_1 \cdot w_1 & \dots & v_1 \cdot w_{m-1} & v_1 \cdot w_m \\ v_2 \cdot w_1 & \dots & v_2 \cdot w_{m-1} & v_2 \cdot w_m \\ \vdots & \ddots & \vdots & \vdots \\ v_{m-1} \cdot w_1 & \dots & v_{m-1} \cdot w_{m-1} & v_{m-1} \cdot w_m \\ w_1 & \dots & w_{m-1} & w_m \end{pmatrix}$$

which is well defined because the vectors $w_1, w_2, \dots, w_m$ occur only in one row, the last. We have that ϕ is a multilinear map. Furthermore, ϕ is a skew-symmetric map on each factor, $(\mathbb{R}^n)^{m-1}$ and $(\mathbb{R}^n)^m$, separately. Hence, ϕ induces a bilinear map

$$\Phi_m : \bigwedge^{m-1}(\mathbb{R}^n) \times \bigwedge^m(\mathbb{R}^n) \longrightarrow \mathbb{R}^n$$

given by

$$\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, w_1 \wedge \cdots \wedge w_m) = \phi(v_1, \ldots, v_{m-1}, w_1, \ldots, w_m),$$

the m-Gram-Schmidt operator (see [11]). Such a family of operators has very nice properties, which we list bellow.

THEOREM 1.1

(1) *The vector* $\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, w_1 \wedge \cdots \wedge w_m)$ *is a linear combination of* $w_1, \ldots, w_m$ *with the coefficient of* w_m *equal to the inner product*

$$(v_1 \wedge \cdots \wedge v_{m-1}) \cdot (w_1 \wedge \cdots \wedge w_{m-1}).$$

(2) *Given* $X \in \mathbb{R}^n$*, the inner product*

$$\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, w_1 \wedge \cdots \wedge w_m) \cdot X$$

equals

$$\det \begin{pmatrix} v_1 \cdot w_1 & \ldots & v_1 \cdot w_{m-1} & v_1 \cdot w_m \\ v_2 \cdot w_1 & \ldots & v_2 \cdot w_{m-1} & v_2 \cdot w_m \\ \vdots & \ddots & \vdots & \vdots \\ v_{m-1} \cdot w_1 & \ldots & v_{m-1} \cdot w_{m-1} & v_{m-1} \cdot w_m \\ w_1 \cdot X & \ldots & w_{m-1} \cdot X & w_m \cdot X \end{pmatrix}.$$

(3) *The vector* $\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, w_1 \wedge \cdots \wedge w_m)$ *is perpendicular to the vectors* $v_1, \ldots, v_{m-1}$.

(4) *Given* m, $2 < m \leq n$, $\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, v_1 \wedge \cdots \wedge v_m)$ *is perpendicular to the vectors*

$$\Phi_j(v_1 \wedge \cdots \wedge v_{j-1}, v_1 \wedge \cdots \wedge v_j),$$

for $2 \leq j < m$.

(5) *The set*

$$\{v_1, \Phi_j(v_1 \wedge \cdots \wedge v_{j-1}, v_1 \wedge \cdots \wedge v_j);\ 2 \leq j \leq m\}$$

is orthogonal.

(6) *Given* m, $2 \leq m \leq n$,

$$\|\Phi_m(v_1 \wedge \cdots \wedge v_{m-1}, v_1 \wedge \cdots \wedge v_m)\| = \|v_1 \wedge \cdots \wedge v_{m-1}\| \, \|v_1 \wedge \cdots \wedge v_m\|.$$

(7) *Given* $1 < j \leq m \leq n$, *define* $W_j = \Phi_j(v_1 \wedge \cdots \wedge v_{j-1}, v_1 \wedge \cdots \wedge v_j)$ *and* $W_1 = v_1$. *If* $\{v_1, \ldots, v_m\}$ *is linearly independent, then*

$$\{V_j = \frac{W_j}{\|W_j\|};\ 1 \leq j \leq m\}$$

is an orthonormal set of vectors, which is an orthonormal basis of $\mathbb{R}^n$, *in the case* $m = n$. *Furthermore,* $W_1 \wedge \cdots \wedge W_j$ *equals*

$$\left(\|v_1\|^2 \|v_1 \wedge v_2\|^2 \cdots \|v_1 \wedge \cdots \wedge v_{j-1}\|^2\right) v_1 \wedge \cdots \wedge v_j$$

and

$$V_1 \wedge \cdots \wedge V_j = \frac{v_1 \wedge \cdots \wedge v_j}{\|v_1 \wedge \cdots \wedge v_j\|}.$$

Whence, if $m = n$, *we get that the bases* $\{v_j;\ 1 \leq j \leq n\}$, $\{W_j;\ 1 \leq j \leq n\}$ *and* $\{V_j;\ 1 \leq j \leq n\}$ *have the same orientation.*

(8) *Suppose that* $\{U_1, \ldots, U_m\}$ *is an orthonormal set of vectors. Then, for each* $1 < j \leq m$,

$$\Phi_j(U_1 \wedge \cdots \wedge U_{j-1}, U_1 \wedge \cdots \wedge U_j) = U_j.$$

(9) *If* $\{e_1, \ldots, e_n\}$ *is the canonical basis of* $\mathbb{R}^n$, *then*

$$\Phi_n(v_1 \wedge \cdots \wedge v_{n-1}, e_1 \wedge \cdots \wedge e_n) = v_1 \times \cdots \times v_{n-1}.$$

The set of orthonormal vectors $\{V_j,\ 1 \leq j \leq m\}$ in (7) is the *Gram-Schmidt orthonormal set* constructed from $\{v_j,\ 1 \leq j \leq m\}$. When $m = n$, we have the *Gram-Schmidt orthonormal basis* constructed from the basis $\{v_j,\ 1 \leq j \leq n\}$. The following Corollary gives us a nonrecursive formula for each element of the Gram-Schmidt set, which follows easily from (6) above.

COROLLARY 1.2 *If* $\{v_j,\ 1 \leq j \leq m\} \subset \mathbb{R}^n$ *is linearly independent, then the elements of the Gram-Schmidt orthonormal set* $\{V_j,\ 1 \leq j \leq m\}$ *are given by* $V_1 = \frac{v_1}{\|v_1\|}$ *and, for* $2 \leq j \leq m$,

$$V_j = \frac{\det\begin{pmatrix} v_1 \cdot v_1 & \ldots & v_1 \cdot v_{j-1} & v_1 \cdot v_j \\ v_2 \cdot v_1 & \ldots & v_2 \cdot v_{j-1} & v_2 \cdot v_j \\ \vdots & \ddots & \vdots & \vdots \\ v_{j-1} \cdot v_1 & \ldots & v_{j-1} \cdot v_{j-1} & v_{j-1} \cdot v_j \\ v_1 & \ldots & v_{j-1} & v_j \end{pmatrix}}{\|v_1 \wedge \cdots \wedge v_{j-1}\| \, \|v_1 \wedge \cdots \wedge v_j\|}.$$

2. The Main Results

Fixed $t \in I$ and given $X \in \mathbb{R}^n$, $1 \leq j \leq n$, and we indicate by $[X]_j$ the j^{th} coordinate of X in the Frenet frame

$$\mathcal{F}(t) = \{V_1(t), V_2(t), \ldots, V_n(t)\},$$

that is

$$X = [X]_1 V_1 + [X]_2 V_2 + [X]_3 V_3 + \cdots + [X]_n V_n.$$

Using the above notation, we have the following result.

THEOREM 2.1 *Let $f : I \longrightarrow \mathbb{R}^n$ be a $(n-1)$-regular parametrized curve in $\mathbb{R}^n$ with speed $\nu = \|f'\|$ and Frenet apparatus*

$$\mathcal{A} = \{\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, V_1, V_2, \ldots, V_n\}.$$

Then

(1) $[f']_1 = \nu, \quad [f^{(m)}]_m = \kappa_1 \ldots \kappa_{m-1}\nu^m, \quad 2 \leq m \leq n;$

(2)

$$f' = \nu V_1$$

and

$$f' \wedge \cdots \wedge f^{(m)} = \nu^{\frac{m(m+1)}{2}} \kappa_1^{m-1} \cdots \kappa_{m-1} V_1 \wedge \cdots \wedge V_m,$$

for $2 \leq m \leq n$.

Proof. We proceed by induction on m. We have

$$[f']_1 = f' \cdot V_1 = \nu V_1 \cdot V_1 = \nu.$$

Since $f^{(m-1)}$ belongs to the space generated by $\{V_1, \ldots, V_{m-1}\}$, we have that $f^{(m-1)} \cdot V_m = 0$, and thus

$$\begin{aligned} [f^{(m)}]_m &= f^{(m)} \cdot V_m = -f^{(m-1)} \cdot V_m' \\ &= -\kappa_1 \kappa_2 \ldots \kappa_{m-2}\nu^{m-1} V_{m-1} \cdot (-\kappa_{m-1}\nu V_{m-1}) \\ &= \kappa_1 \kappa_2 \ldots \kappa_{m-1}\nu^m, \end{aligned}$$

which proofs (i). Now, using $[f^{(m)}]_l = 0$, for $l > m$,

$$f' \wedge \cdots \wedge f^{(m)} = (f' \wedge \cdots \wedge f^{(m-1)}) \wedge f^{(m)}$$
$$= (\nu^{(1+2+\cdots(m-1))}\, \kappa_1^{m-2}\, \kappa_2^{m-3} \cdots \kappa_{m-2}\, V_1 \wedge \cdots \wedge V_{m-1}) \wedge ([f^{(m)}]_m\, V_m)$$
$$= \nu^{(1+2+\cdots+m)}\, \kappa_1^{m-1}\, \kappa_2^{m-2} \ldots \kappa_{m-1}\, V_1 \wedge \cdots \wedge V_m,$$

where, in the last step, we use (i). □

The following corollary establishes a recursive algorithm to calculate all curvatures of f, using only its derivatives.

COROLLARY 2.2

$$\kappa_1 = \frac{\|f' \wedge f''\|}{\nu^3},$$
$$\kappa_m = \frac{\left\|f' \wedge \cdots \wedge f^{(m+1)}\right\|}{\nu^{m+1}\, \kappa_1 \ldots \kappa_{m-1}\, \left\|f' \wedge \cdots \wedge f^{(m)}\right\|}, \tag{7}$$

$2 \leq m \leq n-2$, *and, the last curvature, which has a sign, is*

$$\kappa_{n-1} = \frac{(f' \times \cdots \times f^{(n-1)}) \cdot f^{(n)}}{\nu^n\, \kappa_1 \ldots \kappa_{n-2}\, \left\|f' \wedge \cdots \wedge f^{(n-1)}\right\|}. \tag{8}$$

Furthermore,

$$V_n = \frac{f' \times \cdots \times f^{(n-1)}}{\left\|f' \wedge \cdots \wedge f^{(n-1)}\right\|}. \tag{9}$$

Proof. Taking the norm in Theorem 2.1-(ii), and noting that $V_1 \wedge \cdots \wedge V_m$ and $V_1 \wedge \cdots \wedge V_{m-1}$ are unit vectors, produces

$$\left\|f' \wedge \cdots \wedge f^{(m+1)}\right\| = \nu^{\frac{(m+1)(m+2)}{2}}\, \kappa_1^m\, \kappa_2^{m-1} \cdots \kappa_{m-1}^2\, \kappa_m$$

and

$$\left\|f' \wedge \cdots \wedge f^{(m)}\right\| = \nu^{\frac{m(m+1)}{2}}\, \kappa_1^{m-1}\, \kappa_2^{m-2} \cdots \kappa_{m-1}.$$

Dividing these equations, we get (2.2). These arguments show that (2.2) also applies to $m = n-1$, but in this way we lost the signal of κ_{n-1}, that is, we only obtain $|\kappa_{n-1}|$. For this reason, we rewrite Theorem 2.1-(ii) using the cross product:

$$f' \times \cdots \times f^{(n-1)} = \nu^{\frac{n(n-1)}{2}}\, \kappa_1^{n-2}\, \kappa_2^{n-3} \cdots \kappa_{n-2}\, V_n,$$

for $V_n = V_1 \times \cdots \times V_{n-1}$. Since $\kappa_j > 0$, $1 \leq j \leq n-2$, this last equation implies (9) and

$$(f' \times \cdots \times f^{(n-1)}) \cdot f^{(n)} = (\nu^{\frac{n(n-1)}{2}} \kappa_1^{n-2} \kappa_2^{n-3} \ldots \kappa_{n-2} V_n) \cdot ([f]_n V_n)$$

$$= \nu^{\frac{n(n+1)}{2}} \kappa_1^{n-1} \kappa_2^{n-2} \ldots \kappa_{n-1}$$

$$= \nu^n \kappa_1 \kappa_2 \ldots \kappa_{n-2} \left\| f' \wedge f'' \wedge \cdots \wedge f^{(n-1)} \right\| \kappa_{n-1},$$

which implies (8). □

The next corollaries certainly gives us the more efficient way to get the apparatus of Frenet of f in terms of the derivatives only, in a non-recursive form. Before, we will do a cancellation lemma.

LEMMA 2.3 *Let $p : \mathbb{N} \longrightarrow \mathbb{R}$ be a positive function with $p(0) = 1$ and $p(1) = c$. Now define $q(n) = \frac{p(n-1)p(n+1)}{c\,p^2(n)}$. Then $q(n)$ equals*

$$\frac{1}{c^{n+1} q(1) q(2) \cdots q(n-1)} \frac{p(n+1)}{p(n)}.$$

Proof. Using the definition of q, it easy to check that the factors in $q(1)q(2)\cdots q(n-1)$ cancel nicely resulting $\frac{p(n)}{c^n p(n-1)}$. (Here the reader could use induction on n, observing that $q(1) = p(2)/c^3$.) Hence

$$\frac{1}{c^{n+1} q(1) \cdots q(n-1)} \frac{p(n+1)}{p(n)} = \frac{1}{c^{n+1}} \frac{c^n p(n-1)}{p(n)} \frac{p(n+1)}{p(n)},$$

and the proof is complete. □

COROLLARY 2.4 *Given $2 \leq m \leq n-2$,*

$$\kappa_m = \frac{\left\| f' \wedge \cdots \wedge f^{(m-1)} \right\| \left\| f' \wedge f'' \wedge \cdots \wedge f^{(m+1)} \right\|}{\nu \left\| f' \wedge \cdots \wedge f^{(m)} \right\|^2} \tag{10}$$

and

$$\kappa_{n-1} = \frac{\left\| f' \wedge \cdots \wedge f^{(n-2)} \right\| \left((f' \times \cdots \times f^{(n-1)}) \cdot f^{(n)} \right)}{\nu \left\| f' \wedge \cdots \wedge f^{(n-1)} \right\|^2}. \tag{11}$$

Furthermore, whenever $2 \leq m \leq n-1$,

$$V_m = \frac{\det\begin{pmatrix} f'\cdot f' & \dots & f'\cdot f^{(m-1)} & f'\cdot f^{(m)} \\ f''\cdot f' & \dots & f''\cdot f^{(m-1)} & f''\cdot f^{(m)} \\ \vdots & \ddots & \vdots & \vdots \\ f^{(m-1)}\cdot f' & \dots & f^{(m-1)}\cdot f^{(m-1)} & f^{(m-1)}\cdot f^{(m)} \\ f' & \dots & f^{(m-1)} & f^{(m)} \end{pmatrix}}{\left\|f'\wedge\dots\wedge f^{(m-1)}\right\|\left\|f'\wedge\dots\wedge f^{(m)}\right\|}. \quad (12)$$

Proof. Just set $p(0)=1$, $p(j)=\left\|f'\wedge\dots\wedge f^{(j)}\right\|$, for $j>0$, and use Lemma 2.3 together with Corollary 2.2. The result in (12) follows from the Corollary 1.2, with $v_j = f^{(j)}$, $1 \leq j \leq m$. □

The following corollary, that results easily from the previous results, establishes an algorithm that calculates, using only the derivatives, part of the Frenet apparatus (up to κ_4 and V_3) of f, for any n.

COROLLARY 2.5

(i) $V_1 = \frac{f'}{\nu}$;

(ii) $\kappa_1 = \frac{\|f'\wedge f''\|}{\nu^3}$;

(iii) $V_2 = -\left(\frac{f'\cdot f''}{\nu\|f'\wedge f''\|}\right) f' + \frac{\nu}{\|f'\wedge f''\|} f''$;

(iv) $\kappa_2 = \frac{\|f'\wedge f''\wedge f'''\|}{\|f'\wedge f''\|^2}$;

(v) $k_3 = \frac{\|f'\wedge f''\|\,\|f'\wedge f''\wedge f'''\wedge f''''\|}{\nu\|f'\wedge f''\wedge f'''\|^2}$;

(vi) $k_4 = \frac{\|f'\wedge f''\|\,\|f'\wedge f''\wedge f'''\wedge f''''\wedge f'''''\|}{\nu\|f'\wedge f''\wedge f'''\wedge f''''\|^2}$ or, if $n=5$, $k_4 = \frac{\|f'\wedge f''\wedge f'''\|\,(f'\times f''\times f'''\times f'''')\cdot f'''''}{\nu\|f'\wedge f''\wedge f'''\wedge f''''\|^2}$;

(vii) $V_3 = \frac{(f'''\cdot f'')(f'\cdot f'')-(f'''\cdot f')\|f''\|^2}{\|f'\wedge f''\|\,\|f'\wedge f''\wedge f'''\|} f' + \frac{(f'''\cdot f')(f'\cdot f'')-(f'''\cdot f'')\nu^2}{\|f'\wedge f''\|\,\|f'\wedge f''\wedge f'''\|} f'' + \frac{\|f'\|^2\,\|f''\|^2-(f'\cdot f'')^2}{\|f'\wedge f''\|\,\|f'\wedge f''\wedge f'''\|} f'''$.

COROLLARY 2.6 *If* $\nu = 1$, *that is,* t *is the arc length parameter, then*

(i) $V_1 = f'$;

(ii) $\kappa_1 = \|f''\|$;

(iii) $V_2 = \frac{f''}{\|f''\|}$;

(iv) $\kappa_2 = \frac{\|f'\wedge f''\wedge f'''\|}{\|f''\|^2}$;

(v) $k_3 = \frac{\|f''\| \, \|f' \wedge f'' \wedge f''' \wedge f''''\|}{\|f' \wedge f'' \wedge f'''\|^2}$;

(vi) $k_4 = \frac{\|f' \wedge f''\| \, \|f' \wedge f'' \wedge f''' \wedge f'''' \wedge f'''''\|}{\|f' \wedge f'' \wedge f''' \wedge f''''\|^2}$ or, if $n = 5$, $k_4 = \frac{\|f' \wedge f'' \wedge f'''\| \, (f' \times f'' \times f''' \times f'''') \cdot f'''''}{\|f' \wedge f'' \wedge f''' \wedge f''''\|^2}$;

(vii) $V_3 = \frac{\|f''\|^3}{\|f' \wedge f'' \wedge f'''\|} f' - \frac{f'' \cdot f'''}{\|f' \wedge f'' \wedge f'''\| \|f''\|} f'' + \frac{\|f''\|}{\|f' \wedge f'' \wedge f'''\|} f''''$.

In particular, for $n = 5$, we get more attractive formulas for the results of [14].

Proof. It follows immediately from the anterior corollary by using $\|f' \wedge f''\| = \|f''\|$, which is true, since f' and f'' are orthogonal vectors. □

COROLLARY 2.7 ([10]–4.3 Theorem) *Given a 2-regular curve $f : I \longrightarrow \mathbb{R}^3$, then*

(i) $\mathbf{T} = V_1 = \frac{f'}{\nu}$;

(ii) $\kappa = \kappa_1 = \frac{\|f' \wedge f''\|}{\nu^3}$;

(iii) $\tau = \kappa_2 = \frac{(f' \wedge f'') \cdot f'''}{\|f' \wedge f''\|^2}$;

(iv) $\mathbf{B} = V_3 = \frac{f' \times f''}{\|f' \times f''\|}$;

(v) $\mathbf{N} = V_2 = \mathbf{B} \times \mathbf{T}$.

Proof. The proof is very simple. We just observe that (v) follows from the positive orientation of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. □

EXAMPLE 2.8 Consider $f(t) = (t, t^2, t^3, t^4)$, $t \in \mathbb{R}$. A direct computation givens the matrix of the derivatives of f:

$$(f' \; f'' \; f''' \; f'''') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2t & 2 & 0 & 0 \\ 3t^2 & 6t & 6 & 0 \\ 4t^3 & 12t^2 & 24t & 24 \end{pmatrix}$$

whose determinant is 288. Hence f is 4-regular. The basic objects for the computation of the Frenet apparatus will be obtained below.

[1] $\nu = \|f'\| = \sqrt{1 + 4t^2 + 9t^4 + 16t^6}$;

[2] $\nu' = \frac{2t\left(2 + 9t^2 + 24t^4\right)}{\sqrt{1 + 4t^2 + 9t^4 + 16t^6}}$;

[3] $\|f' \wedge f''\|^2 = \det \begin{pmatrix} f' \cdot f' & f' \cdot f'' \\ f'' \cdot f' & f'' \cdot f'' \end{pmatrix} = \det \begin{pmatrix} 1 + 4t^2 + 9t^4 + 16t^6 & 2t\left(2 + 9t^2 + 24t^4\right) \\ 2t\left(2 + 9t^2 + 24t^4\right) & 4\left(1 + 9t^2 + 36t^4\right) \end{pmatrix}$

$$= 4(1 + 9t^2 + 45t^4 + 64t^6 + 36t^8);$$

[4] $f' \times f'' \times f''' = (-48t^3, 72t^2, -48t, 12)$;

[5] $(f' \times f'' \times f''') \cdot f'''' = 12 \cdot 24 = 288$;

[6] $\|f' \wedge f'' \wedge f'''\|^2 = 144(1 + 16t^2 + 36t^4 + 16t^6)$;

Now, the Frenet apparatus:

[1] $\kappa_1 = \frac{\|f' \wedge f''\|}{\nu^3} = \frac{2\sqrt{1+9t^2+45t^4+64t^6+36t^8}}{(1+4t^2+9t^4+16t^6)^{3/2}}$;

[2] $\kappa_2 = \frac{\|f' \wedge f'' \wedge f'''\|}{\|f' \wedge f''\|^2} = \frac{3\sqrt{1+16t^2+36t^4+16t^6}}{1+9t^2+45t^4+64t^6+36t^8}$;

[3] $\kappa_3 = \frac{\|f' \wedge f''\|(f' \wedge f'' \wedge f''') \cdot f''''}{\nu \|f' \wedge f'' \wedge f'''\|^2} = \frac{4\sqrt{1+9t^2+45t^4+64t^6+36t^8}}{\sqrt{1+4t^2+9t^4+16t^6}(1+16t^2+36t^4+16t^6)}$;

[4] $V_1 = \frac{f'}{\nu} = \frac{1}{\sqrt{1+4t^2+9t^4+16t^6}}(1, 2t, 3t^2, 4t^3)$;

[5] $V_2 = -\left(\frac{f' \cdot f''}{\nu \|f' \wedge f''\|}\right) f' + \frac{\nu}{\|f' \wedge f''\|} f'' = \frac{\begin{pmatrix} -t\left(2 + 9t^2 + 24t^4\right) \\ 1 - 9t^4 - 32t^6 \\ 3t + 6t^3 - 24t^7 \\ 2t^2\left(3 + 8t^2 + 9t^4\right) \end{pmatrix}}{\sqrt{1+4t^2+9t^4+16t^6}\sqrt{1+9t^2+45t^4+64t^6+36t^8}}$;

[6] $V_4 = \frac{f' \times f'' \times f'''}{\|f' \times f'' \times f'''\|} = \frac{\begin{pmatrix} 1 & 0 & 0 & e_1 \\ 2t & 2 & 0 & e_2 \\ 3t^2 & 6t & 6 & e_3 \\ 4t^3 & 12t^2 & 24t & e_4 \end{pmatrix}}{12\sqrt{1+16t^2+36t^4+16t^6}} = \frac{\begin{pmatrix} -4t^3 \\ 6t^2 \\ -4t \\ 1 \end{pmatrix}}{\sqrt{1+16t^2+36t^4+16t^6}}$.

It remains to calculate V_3. For this, we use the positive orientation of the Frenet frame $\{V_1, V_2, V_3, V_4\}$. We have,

$$V_3 = -V_1 \times V_2 \times V_4,$$

since $\det(V_1, V_2, V_4, V_3) = -\det(V_1, V_2, V_3, V_4) = -1$. We omit the explicit result. In particular, the Frenet apparatus at $t = 0$ is given by

$$\mathcal{A}(0) = \{2, 3, 4, (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

3. On the Fundamental Theorem for Curves

In [6], [12] and [8], we found the following theorem, known as *Fundamental Theorem of the Local Theory of Curves.* Its statement could be posed in this way:

Theorem. *Let $k_i(s)$, $i = 1, \dots n-1$, $s \in I$, be smooth functions with $k_i(s) > 0$, $i = 1, \dots n-2$. For a fixed parameter $s_0 \in I$, suppose we have been given a point $q_0 \in \mathbb{R}^n$ as well as an n frame $e_1(0)$, ..., $e_n(0)$. Then there is a unique curve $c : I \longrightarrow \mathbb{R}^n$ parametrized by arc length and satisfying the following three conditions:*

1. $c(s_0) = q_0$,
2. $e_1(0), \dots, e_n(0)$ *is the Frenet frame of c at q_0,*
3. $k_i(s)$, $i = 1, \dots n-1$, *are the curvatures of c.*

A similar theorem appears in [2], on three-dimensional case. The proofs, in both cases, use the general existence and uniqueness theorem for systems of linear differential equations. In that follows, we extend, the cited theorem to arbitrary speed curves in $\mathbb{R}^n$. A complete proof is presented here. The theorem is as follows.

THEOREM 3.1 *Let $\nu(t)$ and $k_j(t)$, $j \in \{1, 2, \dots n-1\}$, $t \in I$, be smooth functions such that, for all $t \in I$, $\nu(t) > 0$ and $k_j(t) > 0$, $j \leq n-2$. Then there exists $f : I \longrightarrow \mathbb{R}^n$ with speed $\nu(t)$ and curvatures $k_j(t)$, $j \in \{1, 2, \dots n-1\}$. Furthermore, if $g : I \longrightarrow \mathbb{R}^n$ is another curve satisfying these conditions, there exists an orientation preserving isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $g = F \circ f$, that is, f and g are congruent curves.*

Before the proof of this theorem, we obtain some preliminary results. Initially, since we are going to work with two curves f and g, we will make a suitable adjustment in the notation: we will put a tilde over each object associated to the curve g. For example, $\widetilde{\nu}$ and $\widetilde{\kappa}_1$ indicate the speed and the first curvature of g whereas $\widetilde{V}_1$ indicates the first vector field of the Frenet frame of g. Thus, given the curves f and g, both defined in I, we have the Frenet apparatus:

$$\mathcal{A}_f = \{\, \kappa_1,\, \kappa_2, \dots,\, \kappa_{n-1},\, V_1,\, V_2, \dots,\, V_n\}$$

and

$$\mathcal{A}_g = \{\, \widetilde{\kappa}_1,\, \widetilde{\kappa}_2, \dots,\, \widetilde{\kappa}_{n-1}, \widetilde{V}_1, \widetilde{V}_2, \dots, \widetilde{V}_n\}.$$

Now, consider $f : I \longrightarrow \mathbb{R}^n$ a parameterized curve in $\mathbb{R}^n$ and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an orientation preserving isometry, that is, $F(X) = SX + X_0$, where S is a orthogonal matrix of determinant 1. Using F, we construct a new curve, namely, $g = F \circ f$, or $g(t) = F(f(t))$, $t \in I$. The next result establishes that in a certain way g inherits the Frenet apparatus of f.

PROPOSITION 3.2 *The speed of g equals speed of f and the Frenet apparatus of $g = F \circ f$ is given by*

$$\mathcal{A}_g = \{\,\kappa_1,\,\kappa_2,\ldots,\,\kappa_{n-1}, S\;V_1, S\;V_2,\ldots, S\;V_n\}$$

In other words, $\widetilde{\kappa}_j = \kappa_j$, $1 \leq j \leq n-1$, and $\widetilde{V}_j = S\,V_j$, $1 \leq j \leq n$.

Proof. Using the chain rule, we get $g'(t) = \mathrm{d}F_{f(t)}(f'(t)) = Sf'(t)$, Hence $\widetilde{\nu}^2 = (Sf') \cdot (Sf') = \nu^2$, because S preserves the inner product, which proof the claim on the speeds. Firstly, we study the curvatures. Using Corollary 2.4 together the norm of a multivector given in (6), we see that given $v_1, \ldots, v_j \in \mathbb{R}^n$, the following holds

$$\begin{aligned} \|Sv_1 \wedge \cdots \wedge Sv_j\| &= \sqrt{\det((Sv_i)\cdot(Sv_j))} \\ &= \sqrt{\det(v_i \cdot v_j)} = \|v_1 \wedge \cdots \wedge v_j\|\,, \end{aligned}$$

again because S preserves the inner product. Thus, for $1 \leq m \leq n-2$,

$$\begin{aligned} \widetilde{\kappa}_m &= \frac{\left\|g' \wedge \cdots \wedge g^{(m-1)}\right\| \, \left\|g' \wedge \cdots \wedge g^{(m+1)}\right\|}{\widetilde{\nu}\,\left\|g' \wedge \cdots \wedge g^{(m)}\right\|^2} \\ &= \frac{\left\|Sf' \wedge \cdots \wedge Sf^{(m-1)}\right\| \, \left\|Sf' \wedge \cdots \wedge Sf^{(m+1)}\right\|}{\nu\,\left\|Sf' \wedge \cdots \wedge Sf^{(m)}\right\|^2} \\ &= \frac{\left\|f' \wedge \cdots \wedge f^{(m-1)}\right\| \, \left\|f' \wedge \cdots \wedge f^{(m+1)}\right\|}{\nu\,\left\|f' \wedge \cdots \wedge f^{(m)}\right\|^2} = \kappa_m, \end{aligned}$$

since $g^{(j)} = Sf^{(j)}$, for all j, by the chain rule. It remains to see the claim about the Frenet frames and the last curvature. From the definitions, we get

$$\widetilde{V}_1 = \frac{g'}{\widetilde{\nu}} = \frac{S\,f'}{\nu} = S\,\frac{f'}{\nu} = S\,V_1.$$

By induction, assuming that $\widetilde{V}_j = S\,V_j$, $1 \leq j \leq n-1$, we show that $\widetilde{V}_{j+1} = S\,V_{j+1}$. In fact, first

$$\begin{aligned}\widetilde{V}_j' = (S\,V_j)' &= S(-\nu\,\kappa_{j-1}\,V_{j-1} + \nu\,\kappa_j\,V_{j+1})\\ &= -\nu\,\kappa_{j-1}\widetilde{V}_{j-1} + \nu\,\kappa_j S\,V_{j+1}.\end{aligned}$$

On the other hand,

$$\widetilde{V}_j' = -\,\widetilde{\nu}\,\widetilde{\kappa}_{j-1}\widetilde{V}_{j-1} + \widetilde{\nu}\,\widetilde{\kappa}_j\widetilde{V}_{j+1} = -\nu\,\kappa_{j-1}\widetilde{V}_{j-1} + \nu\,\kappa_j\widetilde{V}_{j+1}.$$

Hence $\nu\,\kappa_j S\,V_{j+1} = \nu\,\kappa_j\widetilde{V}_{j+1}$ and thus $\widetilde{V}_{j+1} = S\,V_{j+1}$. Note that the preceding arguments could be used for the curvatures. We will make it so for $\widetilde{\kappa}_{n-1}$. Since that $\{\,V_1,\,V_2,\ldots,\,V_n\}$ is a positively oriented frame and $\det S = 1$, it is not hard to see that

$$\begin{aligned}S\,V_n = S(V_1\times\cdots\times V_{n-1}) &= S\,V_1\times\cdots\times S\,V_{n-1}\\ &= \widetilde{V}_1\times\cdots\times\widetilde{V}_{n-1} = \widetilde{V}_n.\end{aligned}$$

Indeed, it is sufficient to note that $S(V_1\times\cdots\times V_{n-1})\cdot\widetilde{V}_j = 0$, for all $1\le j\le n-1$. Differentiating $\widetilde{V}_n = S\,V_n$ yields $\widetilde{\nu}\,\widetilde{\kappa}_{n-1}\widetilde{V}_{n-1} = \nu\,\kappa_{n-1}S\,V_{n-1}$, from which it follows that $\widetilde{\kappa}_{n-1} = \kappa_{n-1}$ and the proof is complete. □

REMARK 3.3 The proof above shows that if F is an orientation reversing isometry, then all works well, except that $\widetilde{V}_n = -S\,V_n$ and $\widetilde{\kappa}_{n-1} = -\,\kappa_{n-1}$.

The converse of the proposition above is true. Its statement is as follows. In it f is as before, having speed ν and Frenet apparatus

$$\mathcal{A}_f = \{\,\kappa_1,\,\kappa_2,\ldots,\,\kappa_{n-1},\,V_1,\,V_2,\ldots,\,V_n\}.$$

We note here that such a result appears in [8] (Theorem 4.11) and its proof he uses another argument.

PROPOSITION 3.4 *Let $h : I \longrightarrow \mathbb{R}^n$ be a parametrized curve with speed $\overline{\nu}$ and Frenet apparatus*

$$\mathcal{A}_h = \{\,\overline{\kappa}_1,\,\overline{\kappa}_2,\ldots,\,\overline{\kappa}_{n-1},\overline{V}_1,\overline{V}_2,\ldots,\overline{V}_n\}.$$

If $\overline{\nu} = \nu$ and $\overline{\kappa}_j = \kappa_j$, $1\le j\le n-1$, then there exists a unique preserving orientation isometry F of $\mathbb{R}^n$ such that $h = F\circ f$.

Proof. For simplicity, suppose that $0\in I$ and $h(0) = f(0) = (0,0,\ldots,0)$. Now, let S be the orthogonal transformation that sends the Frenet frame $\mathcal{F}_f(0)$ to the Frenet frame $\mathcal{F}_h(0)$, that is, $S\,V_j(0) = \overline{V}_j(0)$, $j = 1, 2, \ldots n$. Of course that $\det S = 1$, for these frames are positively oriented. In that follows, we use the ideas of O'Neill [10], in his proof for $n = 3$. Consider $g = S \circ f$ and the real function

$$r(t) = \sum_{j=1}^{n} \widetilde{V}_j(t) \cdot \overline{V}_j(t), \quad t \in I,$$

where, as before, $\mathcal{F}_g = \{\widetilde{V}_j(t) = S\,V_j(t), \quad 1 \le j \le n\}$ is the Frenet frame field of g. Of course that $g(0) = h(0)$, $\mathcal{F}_g(0) = \mathcal{F}_h(0)$, $\widetilde{\nu} = \overline{\nu} = \nu$ and $\overline{\kappa}_j = \widetilde{\kappa}_j = \kappa_j$, for $1 \le j \le n-1$. The main idea now is to show that g matches h. We start observing that $r(0) = n$, since $\mathcal{F}_g(0) = \mathcal{F}_h(0)$. We have that

$$r' = \sum_{j=1}^{n} (\widetilde{V}_j' \cdot \overline{V}_j + \widetilde{V}_j \cdot \overline{V}_j').$$

Claim: $r' = 0$. In fact, firstly, the first and second summands of r' are

$$\widetilde{V}_1' \cdot \overline{V}_1 + \widetilde{V}_1 \cdot \overline{V}_1' = \nu\,\kappa_1 \widetilde{V}_2 \cdot \overline{V}_1 + \nu\,\kappa_1 \widetilde{V}_1 \cdot \overline{V}_2$$

and

$$\widetilde{V}_2' \cdot \overline{V}_2 + \widetilde{V}_2 \cdot \overline{V}_2' = -\nu\,\kappa_1 \widetilde{V}_2 \cdot \overline{V}_1 - \nu\,\kappa_1 \widetilde{V}_1 \cdot \overline{V}_2 + \nu\,\kappa_2 \widetilde{V}_3 \cdot \overline{V}_2 + \nu\,\kappa_2 \widetilde{V}_2 \cdot \overline{V}_3.$$

Hence

$$\sum_{j=1}^{2} (\widetilde{V}_j' \cdot \overline{V}_j + \widetilde{V}_j \cdot \overline{V}_j') = \nu\,\kappa_2 \widetilde{V}_3 \cdot \overline{V}_2 + \nu\,\kappa_2 \widetilde{V}_2 \cdot \overline{V}_3.$$

By induction on m, $m \le n-1$, we get

$$\sum_{j=1}^{m} (\widetilde{V}_j' \cdot \overline{V}_j + \widetilde{V}_j \cdot \overline{V}_j') = \nu\,\kappa_m \widetilde{V}_{m+1} \cdot \overline{V}_m + \nu\,\kappa_m \widetilde{V}_m \cdot \overline{V}_{m+1}.$$

Thus

$$\begin{aligned} r' &= \nu\,\kappa_{n-1} \widetilde{V}_n \cdot \overline{V}_{n-1} + \nu\,\kappa_{n-1} \widetilde{V}_{n-1} \cdot \overline{V}_n + (\widetilde{V}_n' \cdot \overline{V}_n + \widetilde{V}_n \cdot \overline{V}_n') \\ &= \nu\,\kappa_{n-1} \widetilde{V}_n \cdot \overline{V}_{n-1} + \nu\,\kappa_{n-1} \widetilde{V}_{n-1} \cdot \overline{V}_n - \nu\,\kappa_{n-1} \widetilde{V}_n \cdot \overline{V}_{n-1} - \nu\,\kappa_{n-1} \widetilde{V}_{n-1} \cdot \overline{V}_n = 0 \end{aligned}$$

which proofs the claim. So r is a constant function. From $r(0) = n$ we get $r(t) = n$, $t \in I$. Since each summand of r is at most 1, we get that all of them are equal to 1. In particular, $\widetilde{V}_1 = \overline{V}_1$ which implies that $g' = h'$ and, by integration on $[0, t]$, $g = h$, that is, $h = S \circ f$. The general case is obtained by considering $h(t) - h(0)$ and $f(t) - f(0)$ instead of h and f, respectively. From this we conclude that $S(f(t) - f(0)) = h(t) - h(0)$. Hence $g = F \circ h$, where $F(X) = SX + X_0$, $X_0 = h(0) - S(f(0))$. The uniqueness of F follows as in the reference [8]. □

We are almost ready for proving the fundamental theorem (Theorem 3.1). Before, let us summarize some facts on vector matrices.

FACT 3.5 Given the vectors $V_1, \ldots, V_m$ in $\mathbb{R}^n$, we indicate by

$$V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix} = {}^{\mathrm{T}}(V_1\ V_2\ \ldots\ V_m),$$

where ${}^{\mathrm{T}}M$ indicates the transpose of M, the $n \times 1$ vector matrix with elements V_j. V is called a $(m \times 1)$-*vector matrix*. Note that by stacking all coordinates of the vectors V_j (viewed as column vectors), we obtain an $(mn) \times 1$ real matrix or an mn-column vector.

Given a $p \times q$ real matrix A and a $(q \times 1)$-vector matrix $V = {}^{\mathrm{T}}(V_1\ V_2\ \ldots\ V_q)$, the *product* $W = A\,V$ is defined to be the $(q \times 1)$-vector column matrix

$$W = {}^{\mathrm{T}}(W_1\ W_2\ \ldots\ W_q)$$

such that

$$W_i = \sum_{j=1}^{m} a_{ij} V_j,\ a_{ij} \in \mathbb{R},$$

or

$$\begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1q} \\ a_{21} & a_{22} & \ldots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{p}1} & a_{p2} & \ldots & a_{\mathrm{pq}} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_q \end{pmatrix}.$$

Of course, if $V = (V_1\ V_2\ \ldots\ V_p)$ is a $(1 \times p)$-vector matrix, the product VA is a well defined $(1 \times q)$-vector matrix. The following properties hold true. In them, $a \in \mathbb{R}$, C is a $r \times p$ real matrix, A and B are $p \times q$ real matrices and V and W are $(q \times 1)$-vector matrices.

(i) $A\,(aV) = (aA)\,V = a(A\,V)$;

(ii) $(A + B)\,V = A\,V + B\,V$;

(iii) $A(V+W) = A\,V + A\,W$;

(iv) $(CA)V = C(A\,V)$.

Now, consider a linear system of the type $AV = W$, where A, V and W are as above. From this, an interesting exercise arrives: find a usual linear system equivalent to it. It is easy. Just stack the coordinates of the elements of the matrices V and W, obtaining $\widetilde{V}$ and $\widetilde{W}$ in $\mathbb{R}^{qn}$ and replace A by $\widetilde{A}$, of order $pn \times qn$, where $\widetilde{a}_{ij}$ is the block $\widetilde{a}_{ij} = a_{ij}\mathrm{Id}_n$ and Id_n is the $n \times n$ identity matrix. In the Table 1 below, we see an example for $p = 2$, $q = 3$ and $n = 4$, where $V_i = (\,V_{i1},\, V_{i2},\, V_{i3},\, V_{i4}) \in \mathbb{R}^4$.

Another useful product of vector matrices is what uses the inner product in its construction. We will deal only with vector matrices of order either $n \times 1$ or $1 \times n$. For this, let $V = {}^{\mathrm{T}}(V_1\ V_2\ \ldots\ V_n)$ and $W = (W_1\ W_2\ \ldots\ W_n)$. The *dot product* of V by W is the $n \times n$ real matrix

$$V \cdot W = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \cdot (W_1\ \ldots\ W_n) = \begin{pmatrix} V_1 \cdot W_1 & \ldots & V_1 \cdot W_n \\ V_2 \cdot W_1 & \ldots & V_2 \cdot W_n \\ \vdots & \ddots & \vdots \\ V_n \cdot W_1 & \ldots & V_n \cdot W_n \end{pmatrix}.$$

REMARK 3.6 With this new notation, the inner product in (5) becomes

$$(v_1 \wedge \cdots \wedge v_n) \cdot (w_1 \wedge \cdots \wedge w_n) = \det(v \cdot w).$$

Given a $n \times n$ real matrix A, a $(n \times 1)$-vector matrix V and a $(1 \times n)$-vector matrix W, the following hold:

(i) $A\,(V \cdot W) = (A\,V) \cdot W$;

(ii) $(V \cdot W)A = V \cdot (WA)$.

Of course, we have distributive properties for appropriate choices of the vector matrices. Moreover, if V and W depend differentiably on $t \in I$, then

$$(V \cdot W)' = V' \cdot W + V \cdot W'.$$

Finally, using the facts above, we will prove Theorem 3.1. The proof involves four steps, namely:

Table 1. Expanding a vector linear system to an usual linear system

$$
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}
\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}
=
\begin{pmatrix}
a_{11}\begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \end{pmatrix} + a_{12}\begin{pmatrix} V_{21} \\ V_{22} \\ V_{23} \\ V_{24} \end{pmatrix} + a_{13}\begin{pmatrix} V_{31} \\ V_{32} \\ V_{33} \\ V_{34} \end{pmatrix} \\
a_{21}\begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \end{pmatrix} + a_{22}\begin{pmatrix} V_{21} \\ V_{22} \\ V_{23} \\ V_{24} \end{pmatrix} + a_{23}\begin{pmatrix} V_{31} \\ V_{32} \\ V_{33} \\ V_{34} \end{pmatrix}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
a_{11}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} & a_{12}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} & a_{13}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \\
a_{21}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} & a_{22}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} & a_{23}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \\ V_{21} \\ V_{22} \\ V_{23} \\ V_{24} \\ V_{31} \\ V_{32} \\ V_{33} \\ V_{34} \end{pmatrix}
$$

$$
=
\begin{pmatrix}
a_{11} & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} & 0 & 0 & 0 \\
0 & a_{11} & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} & 0 & 0 \\
0 & 0 & a_{11} & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} & 0 \\
0 & 0 & 0 & a_{11} & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} \\
a_{21} & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & a_{23} & 0 & 0 & 0 \\
0 & a_{21} & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & a_{23} & 0 & 0 \\
0 & 0 & a_{21} & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & a_{23} & 0 \\
0 & 0 & 0 & a_{21} & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & a_{23}
\end{pmatrix}
\begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \\ V_{21} \\ V_{22} \\ V_{23} \\ V_{24} \\ V_{31} \\ V_{32} \\ V_{33} \\ V_{34} \end{pmatrix}
=
\begin{pmatrix} W_{11} \\ W_{12} \\ W_{13} \\ W_{14} \\ W_{21} \\ W_{22} \\ W_{23} \\ W_{24} \\ W_{31} \\ W_{32} \\ W_{33} \\ W_{34} \end{pmatrix}
$$

(Step 1) From the given functions ν and κ_j, $1 \leq j \leq n-1$, we construct, based on the Frenet equations, a system of n^2 first order linear differential equations, which we refer as $(\mathcal{F}S)$.

(Step 2) We apply the general existence and uniqueness theorem for systems of linear differential equations to $(\mathcal{F}S)$ and take the solution that satisfies a certain initial condition. Such a solution is an n-vector column matrix $V = {}^{\mathrm{T}}(V_1 \ V_2 \ \dots \ V_n)$ that depends on t.

(Step 3) The existence assertion of the theorem: we verify that $\{V_1(t), V_2(t), \dots, V_n(t)\}$ is actually a orthonormal frame field and, from the vector function $V_1(t)$, we construct a curve f with speed ν, curvatures κ_j, $1 \leq j \leq n-1$, and Frenet frame field $\mathcal{F}_f = \{V_1, V_2, \dots, V_n\}$.

(Step 4) The uniqueness assertion of the theorem: given a curve g with speed ν and curvatures κ_j, $1 \leq j \leq n-1$, there exists an isometry of $\mathbb{R}^n$ such that $g = F \circ f$. This step we have already seen in the Proposition 3.4.

We start writing the Frenet equations (1) of a given f in a matrix form (see (2), for $n = 5$). They become

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \\ \vdots \\ V_{n-2}' \\ V_{n-1}' \\ V_n' \end{pmatrix} = \begin{pmatrix} 0 & \nu\,\kappa_1 & 0 & 0 & \dots & 0 & 0 \\ -\nu\,\kappa_1 & 0 & \nu\,\kappa_2 & 0 & \dots & 0 & 0 \\ 0 & -\nu\,\kappa_2 & 0 & \nu\,\kappa_3 & \dots & 0 & 0 \\ 0 & 0 & -\nu\,\kappa_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \nu\,\kappa_{n-2} & 0 \\ 0 & 0 & 0 & \dots & -\nu\,\kappa_{n-2} & 0 & \nu\,\kappa_{n-1} \\ 0 & 0 & 0 & \dots & 0 & -\nu\,\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{pmatrix},$$

or simply

$$V' = \mathcal{M}\,V, \tag{$\mathcal{F}S$}$$

where $V_1 = \frac{f'}{\nu}$, $V = {}^{\mathrm{T}}(V_1\ V_2\ \dots\ V_n)$ and $V' = {}^{\mathrm{T}}(V_1'\ V_2'\ \dots\ V_n')$ are n-vector column matrices constructed from the Frenet frame of f, and $\mathcal{M}$, called the *Frenet matrix* of f, is the $n \times n$ skew symmetric matrix such that

$$\mathcal{M}_{ij} = V_i' \cdot V_j = \begin{cases} -\nu\,\kappa_{i-1}, & j = i-1 \\ \nu\,\kappa_i, & j = i+1 \\ 0, & j \notin \{i-i, i+1\}. \end{cases},$$

for $V_i' = -\nu\,\kappa_{i-1}\,V_{i-1} + \nu\,\kappa_i\,V_{i+1}$. Of course even without knowing f, we can consider $(\mathcal{F}S)$ on the interval $I \ni 0$, because we can construct $\mathcal{M}$. Hence, we get a first order system of differential equations for the unknown vector functions $V_1, V_2, \dots, V_n$, which, according to the Fact 3.5, can be viewed as a usual system of n^2 first order linear differential equations. Thus we can apply the general existence and uniqueness theorem for systems of linear differential equations to it and achieve a unique set of functions $\{V_1(t), V_2(t), \dots, V_n(t)\}$ that satisfies $(\mathcal{F}S)$ and the initial conditions $V_j(0) = e_j$, $1 \le j \le n$, where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In this way, we go through the steps 1 and 2.

For the step 3, we consider the $n \times n$ matrix function $A(t) = (V_i(t) \cdot V_j(t))$, or

$$A(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_n(t) \end{pmatrix} \cdot (V_1(t)\ V_2(t)\ \dots\ V_n(t)) = V(t) \cdot {}^{\mathrm{T}}V(t).$$

It is convenient to remark that $A(0) = \mathrm{Id}_n$, where Id_n is the $n \times n$ identity matrix. Differentiating A, we get

$$A' = V' \cdot {}^{\mathrm{T}}V + V \cdot {}^{\mathrm{T}}V' = (\mathcal{M}\, V) \cdot {}^{\mathrm{T}}V + V \cdot {}^{\mathrm{T}}(\mathcal{M}\, V)$$
$$= \mathcal{M}(V \cdot {}^{\mathrm{T}}V) + (V \cdot {}^{\mathrm{T}}V)({}^{\mathrm{T}}\mathcal{M}) = \mathcal{M}\, A + A\,({}^{\mathrm{T}}\mathcal{M})$$

Now, note that the expression $\mathcal{M}\, A + A\,({}^{\mathrm{T}}\mathcal{M})$ is linear as function of A. Thus, we have that A is a solution of the linear matrix differential equation

$$X' = \mathcal{M}\, X + {}^{\mathrm{T}}\mathcal{M}\, X$$

with the initial condition $X(0) = \mathrm{Id}_n$. By using vectorization (that is, by stacking columns) of matrices, it is not hard to show that this equation reduces to a system of n^2 first order linear differential equations (For $n = 3$, see the Table 2 below). Hence A is the unique solution of

$$X' = \mathcal{M}\, X + {}^{\mathrm{T}}\mathcal{M}\, X, \quad X(0) = \mathrm{Id}_n.$$

This fact, together with skew symmetry of $\mathcal{M}$, implies $A(t) = \mathrm{Id}_n$, for all $t \in I$, since $X = \mathrm{Id}_n$ satisfies trivially the equation. In fact, $\mathrm{Id}_n' = 0$ and

$$\mathcal{M}\,\mathrm{Id}_n + {}^{\mathrm{T}}\mathcal{M}\,\mathrm{Id}_n = \mathcal{M} + {}^{\mathrm{T}}\mathcal{M} = 0.$$

We conclude that $\{\, V_1(t),\, V_2(t), \ldots,\, V_n(t)\}$ is actually a orthonormal frame field. In reality, it is a positively oriented orthonormal frame field, since it coincides with the canonical basis at $t = 0$ (the $\det$ is a continuous function). It remains only to construct a curve f for attaining the step 3.

In this point, we have a positively oriented orthonormal frame field $\{\, V_1(t),\, V_2(t), \ldots,\, V_n(t)\}$, $t \in I$, that satisfies $(\mathcal{F}S)$. Since V_1 must be the unit tangent vector of f, there is a natural way to choose the curve f, namely,

$$f(t) = \int_0^t \nu(u)\, V_1(u)\, \mathrm{d}u.$$

From this, we get $f' = \nu\, V_1$, or $V_1 = f'/\nu$. For the moment, aiming to set up notations, we suppose that f is a $(n{-}1)$-regular curve with speed ν_f, curvatures κ_{fj}, $1 \le j \le n-1$, and Frenet frame $\{\, V_{fj}, 1 \le j \le n\}$. Hence

(i) $\nu_f = \nu$;

(ii) $V_{f1} = V_1$;

(iii) $f'' = \nu' \, V_1 + \nu^2 \, \kappa_1 \, V_2$;

(iv) $f' \wedge f'' = \nu^3 \, \kappa_1 \, V_1 \wedge V_2$.

The Corolary 2.4, together with (iv), yields

$$\kappa_{f1} = \frac{\|f' \wedge f''\|}{\nu_f^3} = \kappa_1.$$

Using this and (ii), it follows that $V_{f2} = V_2$. In fact, the differentiation of (ii) gives $\kappa_{f1}\nu_f \, V_{f2} = \kappa_1 \nu \, V_2$. Now, a direct calculation shows that

$$f' \wedge f'' \wedge f''' = \nu^6 \, \kappa_1^2 \, \kappa_2 \, V_1 \wedge V_2 \wedge V_3 = \nu_f^6 \, \kappa_{f1}^2 \, \kappa_2 \, V_1 \wedge V_2 \wedge V_3$$

which, together with the Corolary 2.4 and the derivative of $V_{f2} = V_2$, implies $\kappa_{f2} = \kappa_2$ and $V_{f3} = V_3$. By repeating this process inductively (as in the Theorem 2.1), we conclude, for $2 \leq m \leq n$, that

$$f' \wedge \cdots \wedge f^{(m)} = \nu^{\frac{m(m+1)}{2}} \, \kappa_1^{m-1} \, \kappa_2^{m-2} \cdots \, \kappa_{m-1} \, V_1 \wedge \cdots \wedge V_m,$$

which , in particular, shows the regularity of f, $\kappa_{fj} = \kappa_j$, $1 \leq j \leq n-1$, and $V_{fj} = V_j$, for $1 \leq j \leq n$. We are done. $\square$

Table 2. $X' = \mathcal{M}\, X + {}^{\mathrm{T}}\mathcal{M}\, X$, $X = (V_{ij})$, **as an usual system of linear differential equations,** $\mathrm{n} = 3$, $X = {}^{\mathrm{T}}(1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)$ **is the solution that corresponds to** Id_3

$$\begin{pmatrix} 0 & \nu\,\kappa_1 & 0 & \nu\,\kappa_1 & 0 & 0 & 0 & 0 & 0 \\ -\nu\,\kappa_1 & 0 & 0 & 0 & \nu\,\kappa_1 & 0 & \nu\,\kappa_2 & 0 & 0 \\ 0 & 0 & 0 & -\nu\,\kappa_2 & 0 & 0 & 0 & \nu\,\kappa_1 & 0 \\ -\nu\,\kappa_1 & 0 & \nu\,\kappa_2 & 0 & \nu\,\kappa_1 & 0 & 0 & 0 & 0 \\ 0 & -\nu\,\kappa_1 & 0 & -\nu\,\kappa_1 & 0 & \nu\,\kappa_2 & 0 & \nu\,\kappa_2 & 0 \\ 0 & 0 & 0 & 0 & -\nu\,\kappa_2 & 0 & -\nu\,\kappa_1 & 0 & \nu\,\kappa_2 \\ 0 & -\nu\,\kappa_2 & 0 & 0 & 0 & \nu\,\kappa_1 & 0 & 0 & 0 \\ 0 & 0 & -\nu\,\kappa_1 & 0 & -\nu\,\kappa_2 & 0 & 0 & 0 & \nu\,\kappa_2 \\ 0 & 0 & 0 & 0 & 0 & -\nu\,\kappa_2 & 0 & -\nu\,\kappa_2 & 0 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{21} \\ V_{31} \\ V_{12} \\ V_{22} \\ V_{32} \\ V_{13} \\ V_{23} \\ V_{33} \end{pmatrix} = \begin{pmatrix} V_{11}' \\ V_{21}' \\ V_{31}' \\ V_{12}' \\ V_{22}' \\ V_{32}' \\ V_{13}' \\ V_{23}' \\ V_{33}' \end{pmatrix}$$

EXAMPLE 3.7 We will consider the system $(\mathcal{F}S)$, with $n = 3$, $\nu = \sqrt{2}$, $\kappa_1 = \kappa_2 = \frac{1}{\sqrt{2}}$ and initial condition

$$V(0) = \left(\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (-1, 0, 0)\,, \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right). \tag{13}$$

So, we have $\mathcal{M}\,V = V'$:

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix}, \qquad (3-\mathcal{F}S)$$

subject to (13). We will indicate two ways for solving $(3-\mathcal{F}S)$: one for illustrating the conversion to an usual system of linear differential equations as in the Table 1 and the other going hand in hand with an algorithm that plays a key rule in the classification of the curves of constant curvatures. Thus, at the end, we will get a curve with speed $\sqrt{2}$, curvature and torsion equal to $\frac{1}{2}$.

Solution 1. Let $V_i(t) = (V_{i1}(t), V_{i2}(t), V_{i3}(t))$, $i = 1, 2, 3$. Hence $V = (V_1, V_2, V_3)$ is the unknown of the system, which converted to its usual form becomes $\widetilde{\mathcal{M}}\,\widetilde{V} = \widetilde{V}'$:

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \\ V_{13} \\ V_{21} \\ V_{22} \\ V_{23} \\ V_{31} \\ V_{32} \\ V_{33} \end{pmatrix} = \begin{pmatrix} V_{11}' \\ V_{12}' \\ V_{13}' \\ V_{21}' \\ V_{22}' \\ V_{23}' \\ V_{31}' \\ V_{32}' \\ V_{33}' \end{pmatrix}$$

with

$$\widetilde{V}(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1, 0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Remember that the important piece of the solution V is the vector $V_1 = (V_{11}, V_{12}, V_{13})$, from which, by integration of $\nu\,V_1$, we construct the curve f.

It is well known fact that

$$\widetilde{V}(t) = (e^{t\widetilde{\mathcal{M}}})\,\widetilde{V}(0) = \left(\sum_{j=0}^{\infty} \frac{t^j}{j!}\widetilde{\mathcal{M}}^j\right)\widetilde{V}(0)$$

is the unique solution of $(3-\mathcal{F}S)$. An elementary procedure to calculate this solution is indicated next. Patiently, only looking at the first three coordinates of $\widetilde{V}(t)$ and using induction on j, it is possible to obtain that (V_{11}, V_{12}, V_{13}) equals

$$\frac{1}{\sqrt{2}}\left(\sum_{m=0}^{\infty} \frac{(-1)^{m-1}t^{2m+1}}{(2m+1)!}, \quad \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m+1)!}, \quad 1\right).$$

Hence

$$V_1(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1).$$

Now, by integrating $\nu\, V_1$,

$$\int_0^t \nu\, V_1(u)\,\mathrm{d}u = \int_0^t \sqrt{2}\, V_1(u)\,\mathrm{d}u = \int_0^t (-\sin u, \cos u, 1)\,\mathrm{d}u = (\cos t - 1, \sin t, t),$$

which, after a translation, yields $f(t) = (\cos t, \sin t, t)$, the very well known circular helix of $\mathbb{R}^3$, as expected.

Solution 2. Here, we come back to consider the original system $(3 - \mathcal{F}S)$: $\mathcal{M}\, V = V'$, where $V = (\, V_1,\, V_2,\, V_3)$, together with $V(0) = (\, V_1(0),\, V_2(0),\, V_3(0))$, where

$$\begin{cases} V_1(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ V_2(0) = (-1, 0, 0) \\ V_3(0) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{cases}$$

Again, the solution is $V(t) = (e^{t\mathcal{M}})\, V(0)$. We are going to calculate $V(t)$ and then the curve $f(t)$.

(Step 1) To reduce $\mathcal{M}$ to a simpler form, which is possible because it is a skew symmetric matrix. Actually, there exists a orthogonal matrix Q such that $\mathcal{N} = Q^{-1}\mathcal{M}Q$ is a block matrix of the kind

$$\mathcal{N} = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

for some $a \in \mathbb{R}$. This comes from the *normal-form of a skew symmetric matrix theorem*, which can be found in [4].

(Step 2) To calculate $e^{t\mathcal{N}}$, this is easy, because the powers of $\mathcal{N}$ reduce to those of the block A, whose exponential is not hard to get. We have that

$$e^{t\mathcal{N}} = \begin{pmatrix} e^{tA} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos at & -\sin at & 0 \\ \sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Step 3) We solve the new system $\mathcal{N}\,W = W'$, with $W(0) = Q^{-1}\,V(0)$. Of course the solution of this equation is $W = e^{t\mathcal{N}}(Q^{-1}\,V(0))$.

(Step 4) Finally, we obtain the desired solution of $\mathcal{M}\,V = V'$, namely, $V = Q\,W$. In fact, from the previous step, we get $Q\,W(0) = V(0)$ and

$$V' = Q\,W' = Q\,\mathcal{N}\,W = \mathcal{M}\,V.$$

Now the execution of the steps. In the Step 1, according to the proof of the cited theorem, we must consider $M_s = \mathcal{M}^2$ that a symmetric matrix and thus there exist a basis of eigenvectors that diagonalizes it. From this basis we construct the matrix Q in the Step 1. A direct calculation shows that M_s has an double eigenvalue $\lambda_1 = -1$ and the other one $\lambda_2 = 0$. The vectors $v_1 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $v_2 = (0,1,0)$ and $v_3 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ are unit eigenvectors of M_s, v_1 and v_2 are associated to λ_1. (Remember that if $P = (v_1\ v_2\ v_3)$ is the matrix whose columns are the column vectors v_1, v_2 and v_3 then $P^{-1}\,M_s\,P = \operatorname{diag}(-1,-1,0)$). Looking carefully at the proof of the normal-form of a skew symmetric matrix theorem that we cite above, we construct the matrix Q. Its columns are $q_1 = v_1$, $q_2 = \frac{1}{\sqrt{-\lambda 1}}\mathcal{M}\,v_1$ and $q_3 = v_3$. More precisely,

$$Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Hence

$$\mathcal{N} = Q^{-1}\,\mathcal{M}\,Q = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we execute the Step 2 and get

$$e^{t\mathcal{N}} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Then the Step 3 gives us

$$W = e^{t\mathcal{N}}(Q^{-1}\,V(0)) = \begin{pmatrix} \frac{1}{2}\left(\sqrt{2}\,V_2(0)\sin t + (V_1(0) - V_3(0))\cos t + V_1(0) + V_3(0)\right) \\ \frac{(V_3(0)-V_1(0))\sin t}{\sqrt{2}} + V_2(0)\cos t \\ \frac{1}{2}\left(-\sqrt{2}\,V_2(0)\sin t + (V_3(0) - V_1(0))\cos t + V_1(0) + V_3(0)\right) \end{pmatrix}.$$

By substituting $V_1(0)$, $V_2(0)$ and $V_3(0)$ by their values (as row vectors), we obtain

$$W = \begin{pmatrix} \sin t & -\cos t & 0 \\ -\cos t & -\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this, it comes the solution V of $\mathcal{M}V = V'$:

$$V = QW = \begin{pmatrix} -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos(t) & -\sin t & 0 \\ \frac{\sin t}{\sqrt{2}} & -\frac{\cos(t)}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The Steps are all done. By using the first row of V, we get the curve f, exactly as at the end of the Solution 1, that is,

$$\begin{aligned} f(t) &= (1,0,0) + \int_0^t \sqrt{2}\left(-\frac{\sin u}{\sqrt{2}}, \frac{\cos u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \mathrm{d}u \\ &= (\cos t, \sin t, t). \end{aligned}$$

I hope this example, mainly its second solution, helps you in the proof of the theorem on the classification of the curves of constant curvatures that we make in the next section.

4. On Curves of Constant Curvatures

In this section we will use the flowing notation, in order to simplify some calculations involving inner product in $\mathbb{R}^{2m}$.

Given $X = (a_1, b_1, \ldots, a_m, b_m) \in \mathbb{R}^{2m}$, the *complex representation of* X will be indicate by

$$X_{\mathbb{C}} = (a_1 + \mathbf{i}\, b_1, ..., a_m + \mathbf{i}\, b_m) \in \mathbb{C}^m, \quad \mathbf{i} = \sqrt{-1}.$$

With this notation, together with the real part of $z \in \mathbb{C}$, $\Re z$, we get the following useful properties:

(i) $X \cdot Y = \Re(X_{\mathbb{C}} \cdot Y_{\mathbb{C}})$,

(ii) $X \cdot X = X_{\mathbb{C}} \cdot X_{\mathbb{C}}$, *that is,* $\|X\| = \|X_{\mathbb{C}}\|$, where the dot, $\cdot$, indicates the usual inner products in $\mathbb{R}^{2m}$ as well as that[2] in $\mathbb{C}^m$.

4.1. The Even Dimension Case

We start with an example in $\mathbb{R}^4$. Given $t \in \mathbb{R}$, consider $f(t)$ defined by

$$(r_1 \cos(a_1 t), r_1 \sin(a_1 t), r_2 \cos(a_2 t), r_2 \sin(a_2 t)), \tag{14}$$

where a_1, a_2, r_1 and r_2 positive numbers with $a_1 \neq a_2$. The complex representation of f in $\mathbb{C}^2$ is

$$g(t) = (f(t))_{\mathbb{C}} = (r_1 e^{\mathbf{i}\, t a_1}, r_2 e^{\mathbf{i}\, t a_2}).$$

Note that

$$g'(t) = (\mathbf{i}\, r_1 a_1 e^{\mathbf{i}\, t a_1}, \mathbf{i}\, r_2 a_2 e^{\mathbf{i}\, t a_2}),$$

and

$$g''(t) = (-r_1 a_2^2 e^{\mathbf{i}\, t a_1}, -r_2 a_1^2 e^{\mathbf{i}\, t a_2}).$$

More generally,

$$g^{(j)}(t) = (\mathbf{i}^j r_1 a_1^j e^{\mathbf{i}\, t a_1}, \mathbf{i}^j r_2 a_2^j e^{\mathbf{i}\, t a_2}).$$

So, we obtain fast that

$$\nu^2 = \|f'\|^2 = \|g'\|^2 = r_1^2 a_1^2 + r_2^2 a_2^2,$$

$$\|f''\|^2 = \|g''\|^2 = a_1^4 r_1^2 + a_2^4 r_2^2$$

and

$$f' \cdot f'' = \Re(g' \cdot g'') = \Re(-\mathbf{i}\, \left(a_1^3 r_1^2 + a_2^3 r_2^2\right)) = 0.$$

The notable here is that, given any j and k in $\mathbb{N}$, the inner product

$$\begin{aligned} g^{(j)}(t) \cdot g^{(k)}(t) &= (r_1 (\mathbf{i}\, a_1)^j e^{\mathbf{i}\, a_1 t}) r_1 (-\mathbf{i}\, a_1)^k e^{-\mathbf{i}\, a_1 t} + r_2 (\mathbf{i}\, a_2)^j e^{\mathbf{i}\, a_2 t}) r_2 (-\mathbf{i}\, a_2)^k e^{-\mathbf{i}\, a_2 t}) \\ &= \mathbf{i}^j (-\mathbf{i})^k \left(r_1^2 a_1^{j+k} + r_2^2 a_2^{j+k}\right) \\ &= (-1)^k \mathbf{i}^{j+k} \left(r_1^2 a_1^{j+k} + r_2^2 a_2^{j+k}\right) \end{aligned}$$

does not depend on t and thus $f^{(j)}(t) \cdot f^{(k)}(t)$ as well. Thus for any $j \in \mathbb{N}$, $\left\| f' \wedge \cdots \wedge f^{(j)} \right\|$ does not depend on t and then all the curvatures of f must be constant,

[2] $Z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ and $W = (w_1, w_2, \ldots, w_m) \in \mathbb{C}^m$, then

$$Z \cdot W = \sum_{j=1}^{m} z_j \overline{w_j}.$$

according to the Corollary 2.4. Of course it remains to verify that f is at least 3-regular, because without this information, it makes no sense to calculate its curvatures. For this we return to $\mathbb{R}^4$ and write

$$f(t)=\begin{pmatrix}\cos(ta_1) & -\sin(ta_1) & 0 & 0\\ \sin(ta_1) & \cos(ta_1) & 0 & 0\\ 0 & 0 & \cos(ta_2) & -\sin(ta_2)\\ 0 & 0 & \sin(ta_2) & \cos(ta_2)\end{pmatrix}\begin{pmatrix}r_1\\0\\r_2\\0\end{pmatrix} \tag{15}$$

or $f(t)=M(t)f(0)$. Note that $M(t)$ is an one-parameter family of orthogonal matrices of determinant 1. Moreover, we have that $f^{(j)}(t)=M(t)f^{(j)}(0)$, for any $j\in\mathbb{N}$. Collecting this information, for $j\in\{1,2,3,4\}$, in an matrix form, we obtain

$$(f'(t)\ f''(t)\ f'''(t)\ f^{(4)}(t))=M(t)\,(f'(0)\ f''(0)\ f'''(0)\ f^{(4)}(t)).$$

In other words, the matrix $(f'(t)\ f''(t)\ f'''(t)\ f^{(4)}(t))$ equals

$$\begin{pmatrix}\cos(ta_1) & -\sin(ta_1) & 0 & 0\\ \sin(ta_1) & \cos(ta_1) & 0 & 0\\ 0 & 0 & \cos(ta_2) & -\sin(ta_2)\\ 0 & 0 & \sin(ta_2) & \cos(ta_2)\end{pmatrix}\begin{pmatrix}0 & -a_1^2r_1 & 0 & a_1^4r_1\\ a_1r_1 & 0 & -a_1^3r_1 & 0\\ 0 & -a_2^2r_2 & 0 & a_2^4r_2\\ a_2r_2 & 0 & -a_2^3r_2 & 0\end{pmatrix}.$$

Thus the rank of $(f'(t)\ f''(t)\ f'''(t)\ f^{(4)}(t))$ is equal to the rank of $(f'(0)\ f''(0)\ f'''(0)\ f^{(4)}(0))$ that is equal to 4, which can be calculated directly or by using the next Lemma. Now we can calculate the curvatures, by using the Corollary 2.4.

(i) $\kappa_1=\frac{\|f''\|}{\nu^2}=\frac{\sqrt{a_1^4r_1^2+a_2^4r_2^2}}{a_1^2r_1^2+a_2^2r_2^2}$;

(ii) $\kappa_2=\frac{a_2\left(a_1^3-a_1a_2^2\right)r_1r_2}{\left(a_1^2r_1^2+a_2^2r_2^2\right)\sqrt{a_1^4r_1^2+a_2^4r_2^2}}$;

(iii) $\kappa_3=\frac{a_1a_2}{\sqrt{a_1^4r_1^2+a_2^4r_2^2}}$.

As a particular case, consider $r_1=1$, $r_2=\frac{1}{2}$, $a_1=1$, $a_2=3$ and, thus,

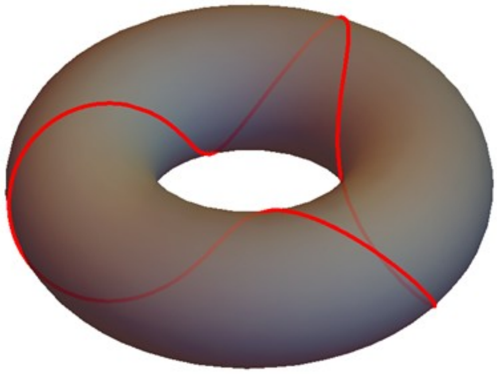
$$f(t)=(\cos(t),\sin(t),\frac{1}{2}\cos(3t),\frac{1}{2}\sin(3t)),\quad t\in\mathbb{R}.$$

We have that the curvatures of f are $\kappa_1=\sqrt{\frac{85}{13}}$, $\kappa_2=-\frac{24}{\sqrt{85}}$ and $\kappa_3=\frac{6}{\sqrt{85}}$. Now, we note that the curve f is contained in the Clifford torus $T_C=S^1(1)\times S^1(\frac{1}{2})\subset S^3(\frac{\sqrt{5}}{2})$, where $S^3(\frac{\sqrt{5}}{2})$ is the tridimensional sphere of radius $\frac{\sqrt{5}}{2}$. By using the stereographic projection

$\pi : S^3(\frac{\sqrt{5}}{2}) - (0,0,0,\frac{\sqrt{5}}{2}) \longrightarrow \mathbb{R}^3,$

$$\pi(x_1, x_2, x_3, xx_4) = \left(\frac{\sqrt{5}x_1}{\sqrt{5} - 2x_4}, \frac{\sqrt{5}x_2}{\sqrt{5} - 2x_4}, \frac{\sqrt{5}x_3}{\sqrt{5} - 2x_4} \right),$$

we can visualize the torus and the curve f in $\mathbb{R}^3$, as in the picture below



In the Theorem 4.3 below, we get, in particular, the converse of this example: If a 3-regular curve in $\mathbb{R}^4$ has constant speed and curvatures, then it is as that in (14), up to an isometry of $\mathbb{R}^4$. To finish, a remark on the angles a_1 and a_2. When they are equal, the trace of f is contained in the intersection of the hyperplanes $\frac{x_3}{r_4} - \frac{x_1}{r_1} = 0$ and $\frac{x_4}{r_4} - \frac{x_2}{r_1} = 0$, which has dimension 2. This would imply that f', f'', f''' are linearly dependent, $k_2 = 0$ and κ_3 is not defined. Thus the condition $a_1 \neq a_2$ guarantees that the curve f is 4-regular.

LEMMA 4.1 *Given m distinct real numbers $a_1, \ldots, a_m$ and any other real number b, define $D(a_1, a_2, \ldots, a_m)$ and $\widetilde{D}(a_1, \ldots, a_m, b)$ as in the Table 3 below. Then*

(i) $\det D(a_1, a_2, \ldots, a_m) = \left(\prod_{j=1}^m a_j^3\right) \prod_{i<j}^m \left(a_i^2 - a_j^2\right)^2.$

(ii) $\det \widetilde{D}(a_1, \ldots, a_m, b) = \left(\prod_{j=1}^m a_j^5\right) \prod_{i<j}^m \left(a_i^2 - a_j^2\right)^2.$

Proof. We use induction on m. The idea of this proof is similar to that used in the calculation of the Vandermonde determinant. Differently, here, the induction hypotheses is attained after two steps. In fact, we start writing $D = D(a_1, a_2, \ldots, a_m) = (D_1, D_2, \ldots, D_{2m})$, where D_j denotes the j^{th} row of D and then by replacing a_1^2 by λ in the first row of D to construct the polynomial

$$p(\lambda) = \det((0, -\lambda, 0, \lambda^2, 0, \ldots, (-1)^m \lambda^m), D_2, \ldots, D_{2m}).$$

Table 3. Matrices for Lema 4.1

$$D(a_1,a_2,\ldots,a_m)=\begin{pmatrix} 0 & -a_1^2 & 0 & a_1^4 & \ldots & 0 & (-1)^m a_1^{2m} \\ a_1 & 0 & -a_1^3 & 0 & \ldots & (-1)^{m-1}a_1^{2m-1} & 0 \\ 0 & -a_2^2 & 0 & a_2^4 & \ldots & 0 & (-1)^m a_2^{2m} \\ a_2 & 0 & -a_2^3 & 0 & \ldots & (-1)^{m-1}a_2^{2m-1} & 0 \\ 0 & -a_3^2 & 0 & a_3^4 & \ldots & 0 & (-1)^m a_3^{2m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -a_m^2 & 0 & a_m^4 & \ldots & 0 & (-1)^m a_m^{2m} \\ a_m & 0 & -a_m^3 & 0 & \ldots & (-1)^{m-1}a_m^{2m-1} & 0 \end{pmatrix}.$$

$$\widetilde{D}(a_1,\ldots,a_m,b)=\begin{pmatrix} 0 & -a_1^2 & 0 & a_1^4 & \ldots & 0 & (-1)^m a_1^{2m} & 0 \\ a_1 & 0 & -a_1^3 & 0 & \ldots & (-1)^{m-1}a_1^{2m-1} & 0 & (-1)^m a_1^{2m+1} \\ 0 & -a_2^2 & 0 & a_2^4 & \ldots & 0 & (-1)^m a_2^{2m} & 0 \\ a_2 & 0 & -a_2^3 & 0 & \ldots & (-1)^{m-1}a_2^{2m-1} & 0 & (-1)^m a_2^{2m+1} \\ 0 & -a_3^2 & 0 & a_3^4 & \ldots & 0 & (-1)^m a_3^{2m} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -a_m^2 & 0 & a_m^4 & \ldots & 0 & (-1)^m a_m^{2m} & 0 \\ a_m & 0 & -a_m^3 & 0 & \ldots & (-1)^{m-1}a_m^{2m-1} & 0 & (-1)^m a_m^{2m+1} \\ b & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

Hence, p either is zero or has degree at most m. Since a matrix with either one zero row or two equal rows has zero determinant, we get that p vanishes at 0 and a_j^2, for each $2\leq j\leq m$. We claim that the degree of p is exactly m. In effect, the coefficient of λ^m is

$$c_m=(-1)^{1+2m}(-1)^m\,M_{1(2m)}=(-1)^{m+1}\,M_{1(2m)},$$

where $M_{1(2m)}$ is the $(1,2m)$-minor of D. By factoring out a_1, we get

$$M_{1(2m)}=a_1\det\begin{pmatrix} 1 & 0 & -a_1^2 & 0 & \ldots & (-1)^{m-1}a_1^{2m-2} \\ 0 & -a_2^2 & 0 & a_2^4 & \ldots & 0 \\ a_2 & 0 & -a_2^3 & 0 & \ldots & (-1)^{m-1}a_2^{2m-1} \\ 0 & -a_3^2 & 0 & a_3^4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_m^2 & 0 & a_m^4 & \ldots & 0 \\ a_m & 0 & -a_m^3 & 0 & \ldots & (-1)^{m-1}a_m^{2m-1} \end{pmatrix}.$$

Denoting the $(2m-1)\times(2m-1)$ matrix above by $B=(B_1,B_2,\ldots,B_{2m-1})$, we define

$q(\lambda)$ to be

$$\det((1, 0, -\lambda, 0, \ldots, (-1)^{m-1}\lambda^{m-1}), B_2, \ldots, B_{2m-1}).$$

Note that we have substituted a_1^2 by λ in the first row of B. The induction hypotheses guarantees that the polynomial q has exactly degree $m-1$ because the coefficient of λ^{m-1} equals

$$\begin{aligned} d_{m-1} &= (-1)^{1+2m-1}(-1)^{m-1}\det D(a_2, a_3, \ldots, a_m) \\ &= (-1)^{m-1}\det D(a_2, a_3, \ldots, a_m) \neq 0. \end{aligned}$$

Looking back in the polynomial p, we see that

$$c_m = (-1)^{m+1}\, a_1\, q(a_1^2) \neq 0,$$

because $\partial q = m-1$ and q vanishes at a_2^2, a_3^2, $\ldots$, a_m^2 that are $m-1$ distinct numbers. Now, we can factor p as

$$p(\lambda) = c_m\, \lambda(\lambda - a_2^2)(\lambda - a_3^2)\cdots(\lambda - a_m^2) = (-1)^{m+1}\, a_1\, q(a_1^2)\, \lambda(\lambda - a_2^2)(\lambda - a_3^2)\cdots(\lambda - a_m^2),$$

which implies

$$\begin{aligned} \det D = p(a_1^2) &= (-1)^{m+1}\, a_1\, q(a_1)\, a_1^2 \prod_{j=2}^{m}(a_1^2 - a_j^2) \\ &= (-1)^{m+1}\, a_1^3\, q(a_1^2) \prod_{j=2}^{m}(a_1^2 - a_j^2). \end{aligned} \tag{16}$$

Again from the induction hypothesis, we obtain that

$$\begin{aligned} q(\lambda) &= d_{m-1}(\lambda - a_2^2)(\lambda - a_3^2)\cdots(\lambda - a_m^2) \\ &= (-1)^{m+1}\det D(a_2, a_3, \ldots, a_m) \prod_{j=2}^{m}(\lambda - a_j^2) \\ &= (-1)^{m+1}\left(\prod_{j=2}^{m} a_j^3\right)\left(\prod_{2\leq i<j}^{m}(a_i^2 - a_j^2)^2\right)\prod_{j=2}^{m}(\lambda - a_j^2), \end{aligned}$$

whence, we obtain $q(a_1^2)$ equal to

$$(-1)^{m+1}\left(\prod_{j=2}^{m} a_j^3\right)\left(\prod_{2\leq i<j}^{m}(a_i^2 - a_j^2)^2\right)\prod_{j=2}^{m}(a_1^2 - a_j^2)$$

which substituted in (16) yields finally

$$\det D = \left(\prod_{j=1}^{m} a_j^3\right) \prod_{i<j}^{m} \left(a_i^2 - a_j^2\right)^2 .$$

The result in (ii) follows easily from (i), completing the proof. $\square$

REMARK 4.2 The element d_{ij} of $D(a_1, a_2, \dots, a_m)$ equals $f^{(j)}(0) \cdot e_i$, where e_i is the i^{th} vector of the canonical basis of the $\mathbb{R}^{2m}$ and f is as in the following classification theorem.

THEOREM 4.3 *Let $\nu > 0$ and κ_j, $1 \le j \le n-1$, be constants such that $\kappa_j > 0$, $1 \le j \le n-2$, and $\kappa_{n-1} \neq 0$. Let f be a curve with speed ν and curvatures κ_j, $1 \le j \le n-1$. Suppose that n is even. Then there exist positives real numbers a_j and r_j, $1 \le j \le m$, $m = \frac{n}{2}$, such that, up to an isometry,*

$$f(t) = (r_1 e^{\mathbf{i}\,t\,a_1}, r_2 e^{\mathbf{i}\,t\,a_2}, \dots, r_m e^{\mathbf{i}\,t\,a_m}),$$

where $e^{\mathbf{i}\,t\,a_j} = \cos t a_j + \mathbf{i}\,\sin t a_j$.

Proof. Some of the ideas in this proof are suggested in [6] (**2.16-Remark**). We give a full proof for $n = 4$ and believe that its extension to the general case is very easy.

Let $\widetilde{f}$ be the 3-regular curve obtained from the unique solution

$$V = (\,V_1,\, V_2,\, V_3,\, V_4)$$

of the linear system $\mathcal{M}\, V = V'$ or, more explicitly,

$$\begin{pmatrix} 0 & \nu\kappa_1 & 0 & 0 \\ -\nu\kappa_1 & 0 & \nu\kappa_2 & 0 \\ 0 & -\nu\kappa_2 & 0 & \nu\kappa_3 \\ 0 & 0 & -\nu\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{pmatrix},$$

with the initial condition $V(0) = (Q_1, Q_2, Q_3, Q_4)$, where $\{Q_1, Q_2, Q_3, Q_4\}$ will be an orthonormal basis defined as follows. The existence of the curve $\widetilde{f}$ is guaranteed by the Theorem 3.1. Remember, $\widetilde{f}(t) = \int_0^t \nu\, V_1(u)\mathrm{d}u$. Since $\mathcal{M}$ is skew symmetric, we obtain from the normal-form of a skew symmetric matrix theorem an orthonormal basis

$\{Q_1, Q_2, Q_3, Q_4\}$ that reduces $\mathcal{M}$ to

$$\mathcal{N} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -a_2 & 0 \end{pmatrix},$$

where $0 < a_1$ and $0 < a_2$. Actually, $\mathcal{N} = {}^{\mathrm{T}}Q\,\mathcal{M}\,Q$, where Q is the 4×4 matrix whose columns are the vectors Q_1, Q_2, Q_3, Q_4 viewed as column vectors. By rewriting the equation $\mathcal{M}\,V = V'$, with $V(0) = (Q_1, Q_2, Q_3, Q_4)$, in terms of $\mathcal{N}$, we get $(Q\,\mathcal{N}\,{}^{\mathrm{T}}Q)\,V = V'$, which is the same as

$$\mathcal{N}\,{}^{\mathrm{T}}(Q\,V) = {}^{\mathrm{T}}Q\,V' = ({}^{\mathrm{T}}Q\,V)',$$

subject to

$$({}^{\mathrm{T}}Q\,V)(0) = E = (e_1, e_2, e_3, e_4),$$

where $\{e_1, e_2, e_3, e_4\}$ is the canonical basis of $\mathbb{R}^4$. By the uniqueness of solution of the equation above, we must have $Q\,V = e^{t\mathcal{N}}\,E$ or $V = {}^{\mathrm{T}}Q\,e^{t\mathcal{N}}\,E$, where, explicitly,

$${}^{\mathrm{T}}Q = \begin{pmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ q_{12} & q_{22} & q_{32} & q_{42} \\ q_{13} & q_{23} & q_{33} & q_{43} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix},$$

$$e^{t\mathcal{N}} = \begin{pmatrix} \cos(a_1 t) & \sin(a_1 t) & 0 & 0 \\ -\sin(a_1 t) & \cos(a_1 t) & 0 & 0 \\ 0 & 0 & \cos(a_2 t) & \sin(a_2 t) \\ 0 & 0 & -\sin(a_2 t) & \cos(a_2 t) \end{pmatrix}$$

and

$$E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

The fours first rows of the product $V = {}^{\mathrm{T}}Q\,e^{t\mathcal{N}}\,E$ is

$$V_1(t) = \begin{pmatrix} q_{11}\cos(a_1 t) - q_{21}\sin(a_1 t) \\ q_{11}\sin(a_1 t) + q_{21}\cos(a_1 t) \\ q_{31}\cos(a_2 t) - q_{41}\sin(a_2 t) \\ q_{31}\sin(a_2 t) + q_{41}\cos(a_2 t) \end{pmatrix} = \begin{pmatrix} \sqrt{q_{11}^2 + q_{21}^2}\,(-\sin(a_1 t - c_1)) \\ \sqrt{q_{11}^2 + q_{21}^2}\cos(a_1 t - c_1) \\ \sqrt{q_{31}^2 + q_{41}^2}\,(-\sin(a_2 t - c_2)) \\ \sqrt{q_{31}^2 + q_{41}^2}\cos(a_2 t - c_2) \end{pmatrix}$$

where c_1 and c_2 are such that

$$\sin c_1 = \frac{q_{11}}{\sqrt{q_{11}^2 + q_{21}^2}}, \ \cos c_1 = \frac{q_{21}}{\sqrt{q_{11}^2 + q_{21}^2}},$$

$$\sin c_2 = \frac{q_{31}}{\sqrt{q_{31}^2 + q_{41}^2}}, \ \cos c_2 = \frac{q_{41}}{\sqrt{q_{31}^2 + q_{41}^2}}.$$

Note that V_1 is in fact an unit vector field. Integration of νV_1 yields

$$\widetilde{f}(t) = \int \nu V_1(t)\mathrm{d}t = \nu \begin{pmatrix} \frac{\sqrt{q_{11}^2+q_{21}^2}}{a_1} \cos(a_1 t - c_1) \\ \frac{\sqrt{q_{11}^2+q_{21}^2}}{a_1} \sin(a_1 t - c_1) \\ \frac{\sqrt{q_{31}^2+q_{41}^2}}{a_2} \cos(a_2 t - c_2) \\ \frac{\sqrt{q_{31}^2+q_{41}^2}}{a_2} \sin(a_2 t - c_2)) \end{pmatrix} = \begin{pmatrix} r_1 \cos(a_1 t - c_1) \\ r_1 \sin(a_1 t - c_1) \\ r_2 \cos(a_2 t - c_2) \\ r_2 \sin(a_2 t - c_2)) \end{pmatrix}.$$

Since we know that $\widetilde{f}$ is 3-regular, it follows that $a_1 \neq a_2$, according to the remark that we did at the end of the example above. Furthermore, $\nu_{\widetilde{f}} = \nu$ and $\widetilde{\kappa}_j = \kappa_j$, $1 \leq j \leq 3$. A simple computation shows that the curve f obtained from $\widetilde{f}$ by taking $c_1 = c_2 = 0$ has also the speed ν and curvatures κ_j, $1 \leq j \leq 3$, for the inner products involving its derivatives are exactly the same as those of $\widetilde{f}$. We are done. □

REMARK 4.4 In [1] and [12], we find an approach involving one-parameter subgroup of isometries to perform the calculations of the curvatures as well as to prove the constant curvature classification theorem. Here, in (15), we have one example of such a subgroup, namely, $M(t)$, since $M(t_1 + t_2) = M(t_1)M(t_2)$.

4.2. The Odd Dimension Case

Consider the following generalization of the circular helix $f(t)$ given by

$$(r_1 \cos(a_1 t), r_1 \sin(a_1 t), r_2 \cos(a_2 t), r_2 \sin(a_2 t), bt),$$

$t \in \mathbb{R}$, where a_1, a_2, r_1 and r_2 positive numbers with $a_1 \neq a_2$ and $b \neq 0$. The complex representation of f in $\mathbb{C}^3$ is

$$g(t) = (f(t))_{\mathbb{C}} = (r_1 e^{\mathbf{i}\,ta_1}, r_2 e^{\mathbf{i}\,ta_2}, bt),$$

Note that $g'(t) = (\mathbf{i}\,r_1 a_1 e^{\mathbf{i}\,ta_1}, \mathbf{i}\,r_2 a_2 e^{\mathbf{i}\,ta_2}, b)$, $g''(t) = (-r_1 a_1^2 e^{\mathbf{i}\,ta_1}, -r_2 a_1^2 e^{\mathbf{i}\,ta_2}, 0)$. More generally,

$$g^{(j)}(t) = (\mathbf{i}^j r_1 a_1^j e^{\mathbf{i}\,ta_1}, \mathbf{i}^j r_2 a_1^j e^{\mathbf{i}\,ta_2}, 0),$$

whenever $j > 1$. So

$$\nu^2 = \|f'\|^2 = \|g'\|^2 = r_1^2 a_1^2 + r_2^2 a_2^2 + b^2, \quad \|f''\|^2 = \|g''\|^2 = a_1^4 r_1^2 + a_2^4 r_2^2$$

and

$$f' \cdot f'' = \Re(g' \cdot g'') = \Re(-\mathbf{i}\ \left(a_1^3 r_1^2 + a_2^3 r_2^2\right)) = 0.$$

As before, we get that $f^{(j)}(t) \cdot f^{(k)}(t) = \Re(g^{(j)}(t) \cdot g^{(k)}(t))$ does not depend on t and thus all of the curvatures of f must be constant. Of course it remains to verify that f is at least 3-regular, because without this information, it makes no sense to calculate its curvatures. For this we return to $\mathbb{R}^4$ and write $f(t) = N(t)f(0) + (0, 0, 0, 0, b\,t)$, where

$$N(t) = \begin{pmatrix} \cos(ta_1) & -\sin(ta_1) & 0 & 0 & 0 \\ \sin(ta_1) & \cos(ta_1) & 0 & 0 & 0 \\ 0 & 0 & \cos(ta_2) & -\sin(ta_2) & 0 \\ 0 & 0 & \sin(ta_2) & \cos(ta_2) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $N(t)$ is also a family of one-parameter of orthogonal matrices of determinant 1. By collecting the derivatives $f^{(j)}(t)$, for $j \in \{1, 2, 3, 4, 5\}$, in an matrix, we obtain that $(f'(t)\ f''(t)\ f'''(t)\ f^{(4)}(t)\ f^{(5)}(t))$ is equal to the product

$$N(t) \begin{pmatrix} 0 & -a_1^2 r_1 & 0 & a_1^4 r_1 & 0 \\ a_1 r_1 & 0 & -a_1^3 r_1 & 0 & a_1^5 r_1 \\ 0 & -a_2^2 r_2 & 0 & a_2^4 r_2 & 0 \\ a_2 r_2 & 0 & -a_2^3 r_2 & 0 & a_2^5 r_2 \\ b & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lemma 4.1 implies that the rank of

$$(f'(t)\ f''(t)\ f'''(t)\ f^{(4)}(t)\ f^{(5)}(t))$$

is equal to 5. Hence f is 5-regular and then we can calculate its curvatures, by using the Corollary 2.4.

(i) $\kappa_1 = \frac{\|f''\|}{\nu^2} = \frac{\sqrt{a_1^4 r_1^2 + a_2^4 r_2^2}}{a_1^2 r_1^2 + a_2^2 r_2^2}$;

(ii) $\kappa_2 = \frac{a_2\left(a_1^3 - a_1 a_2^2\right) r_1 r_2}{\left(a_1^2 r_1^2 + a_2^2 r_2^2\right)\sqrt{a_1^4 r_1^2 + a_2^4 r_2^2}}$;

(iii) $\kappa_3 = \frac{a_1 a_2}{\sqrt{a_1^4 r_1^2 + a_2^4 r_2^2}}$.

Now the theorem for the odd dimension case.

THEOREM 4.5 *Let $\nu > 0$ and κ_j, $1 \leq j \leq n-1$, be constants such that $\kappa_j > 0$, $1 \leq j \leq n-2$, and $\kappa_{n-1} \neq 0$. Let f be a curve with speed ν and curvatures κ_j, $1 \leq j \leq n-1$. Suppose that n is odd. Then there exist positives real numbers a_j and r_j, $1 \leq j \leq m$, $m = \frac{n-1}{2}$, and $b \neq 0$, such that, up to an isometry,*

$$f(t) = (r_1 e^{\mathbf{i}\,t\,a_1}, r_2 e^{\mathbf{i}\,t\,a_2}, \ldots, r_m e^{\mathbf{i}\,t\,a_m}, b\,t).$$

Proof. It is similar to that of the Theorem 4.3. The only difference is that the matrix $\mathcal{N}$ is now

$$\mathcal{N} = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

whence,

$$e^{t\mathcal{N}} = \begin{pmatrix} \cos(a_1 t) & \sin(a_1 t) & 0 & 0 & 0 \\ -\sin(a_1 t) & \cos(a_1 t) & 0 & 0 & 0 \\ 0 & 0 & \cos(a_2 t) & \sin(a_2 t) & 0 \\ 0 & 0 & -\sin(a_2 t) & \cos(a_2 t) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This gives us the following curve $f(t)$:

$$(r_1 \cos(a_1 t), r_1 \sin(a_1 t), r_2 \cos(a_2 t), r_2 \sin(a_2 t), bt).$$

Of course, there exist other possibilities for $\mathcal{N}$, for instance, $\mathcal{N}$ could be

$$\mathcal{N} = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which would lead to a curve of the kind

$$f(t) = (r_1 \cos(a_1 t), r_1 \sin(a_1 t), b_1\,t, b_2\,t, b_3\,t),$$

which has $\kappa_3 = 0$ and undefined κ_4. Similarly, any possibility other than $\mathcal{N}$ in (17) is discarded. Of course, we have considered $n = 5$. The general case is almost as this. □

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