Latin American Journal of Mathematics (ISSN 2965-0798)
https://doi.org/10.14244/lajm.v4i1.33

Some Tridiagonal Matrices of the Repunit Sequence *

Eudes Antonio Costa[1], **Douglas Catulio Santos**[2], **Paula M. M. C. Catarino**[3]

[1]Department of Mathematics, Federal University of Tocantins, 77330-000
Arraias, Brazil

[2]Education Department of the State of Bahia, 41745-004
Barreiras, Brazil

[3]Department of Mathematics, University of Trás-os-Montes and Alto Douro, 5000-801
Vila Real, Portugal

eudes@uft.edu.br, catuliodouglas4@gmail.com, pcatarin@utad.pt

Abstract. *This paper explores the connection between tridiagonal matrices and the repunit sequence, which is a type of Horadam sequence, and aims to establish new representations of repunit sequences using distinct tridiagonal matrices and their determinants. Motivated by earlier work relating tridiagonal matrices to second-order linear recurrences, we present another representation of the repunit sequence by tridiagonal matrices.*

Keywords – *Horadam sequence, Repunit sequence, Tridiagonal matrices.*
MSC2020 – *11B37, 11B39, 11B83*

1. Introduction

The repunit numbers $\{r_n\}_{n\geq 0}$ are the terms of the sequence $\{0, 1, 11, 111, \ldots\}$, where each term satisfies the recursive formula $r_{n+1} = 10r_n + 1$ for all $n \geq 0$ and $r_0 = 0$, the sequence $A002275$ in OEIS[1]. In [2], the authors noted that this sequence also satisfies the recurrence relation $r_{n+1} = 11r_n - 10r_{n-1}$, with initial condition $r_0 = 0$, $r_1 = 1$ and $n \geq 1$.

For n natural, consider $\{h_n\}$ the Horadam sequence defined by the second order recurrence relation, where p and q are fixed integers, such that

$$h_{n+1} = ph_n + qh_{n-1},\ n \geq 1\ ,$$

*Received 28 October 2024; received in revised form 23 December 2024; accepted 20 January 2025. Available online 23 January 2025

with initial conditions $h_0 = a$ and $h_1 = b$. This sequence was introduced by Horadam [3, 4], and it generalizes many sequences with the characteristic recurrence relation of the form $x^2 - px + q = 0$. More general results about the Horadam sequence can be found in [3, 5]. So if we let $p = 11; q = -10; a = 0$; and $b = 1$; then the Horadam sequence is specified in the repunit sequence. Other works explore the connections of the repunit sequence with the Lucas-type sequence, another Horadam-type sequence, see [6, 7].

A tridiagonal matrix is a special kind of square matrix in which all elements are zero except those on the main diagonal, the diagonal above the main diagonal (superdiagonal), and the diagonal below the main diagonal (subdiagonal). For example, a 4×4 tridiagonal matrix might look like this:

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 \\ 0 & c_2 & a_3 & b_3 \\ 0 & 0 & c_3 & a_4 \end{bmatrix},$$

where a_1, a_2, a_3, a_4 are the elements of the main diagonal, b_1, b_2, b_3 are the elements of the superdiagonal, and c_1, c_2, c_3 are the elements of the subdiagonal. The importance of this type of study for a particular sequence is that it is easy to express the determinant of this type of matrix.

The structure of this article is organized as follows. In Section 2, we revisit the Binet formula for the repunit sequence, applicable to any integer index, and examine the recurrence relation of this sequence, which is a specific case of the Horadam sequence. In Section 3, we introduce a type of tridiagonal matrix, showing that, by specifying certain input values, the determinant of this matrix generates the repunit sequence, thereby illustrating a practical application. This discussion is further extended to two additional types of tridiagonal matrices in the subsequent sections.

Although the repunit sequence is a particular sequence of Horadam-type, in this work we use tridiagonal matrices to represent this sequence, a subject that is still little explored in mathematical literature.

2. Repunit numbers and Binet formula

Note that the difference equation associated with the sequence of repunit $\{r_n\}_{n\geq 0}$ is

$$r_{n+1} = 11r_n - 10r_{n-1}\,, r_0 = 0 \text{ and } r_1 = 1\,. \tag{1}$$

The characteristics equation for the second order linear difference Equation (1) is given by $x^2 - 11x + 10 = 0$ and its real roots are $x_1 = 10$ and $x_2 = 1$. And from the theory of difference equation we know that the general term of the Equation (1) can be expressed as:

$$r_n = c_1(x_1)^n + c_2(x_2)^n,$$

where c_1 and c_2 are arbitrary constants (to be evaluated) and x_1 and x_2 are characteristics roots. We find $c_1 = \frac{1}{9}$ and $c_2 = -\frac{1}{9}$. So we have that the Binet's formula, for all $n \in \mathbb{N}$:

$$r_n = \frac{10^n - 1}{9} \,. \tag{2}$$

The Equation (2) presents the classic and well-known Binet's formula for the sequence of repunit $\{r_n\}_{n\geq 0}$, a formula to calculate the n-th term of the repunit sequence, see the references [8, 9, 10] .

The repunit sequence are also extendable in the negative direction which can be achieved by rearranging Equation (1). It is also noted that

$$r_{-n} = -\frac{r_n}{10^n} \text{ for all } n \geq 0.$$

It follows from the definition that repunit sequence with negative index is the set of elements given by

$$\{r_{-n}\}_{n\geq 1} = \left\{-\frac{1}{10}, -\frac{11}{10^2}, -\frac{111}{10^3}, \ldots, \right\} = \{-0.1;\ -0.11;\ -0.111, \ldots, \} \,.$$

The first few repunit numbers with negative subscript are given in the following Table 1, with $-8 \leq n \leq -1$:

n	-1	-2	-3	-4	-5	-6	-7	-8
r_n	- 0.1	-0.11	-0.111	-0.1111	-0.11111	-0.111111	-0.1111111	-0.11111111

Table 1. Repunit numbers at negative index [11]

According [11], observation of Table 1, the repunit sequence with negative index $\{r_{-n}\}_{n\geq 1}$ satisfies the recurrence relation

$$r_{-(n+1)} = \frac{11}{10} r_{-n} - \frac{1}{10} r_{-(n-1)} \text{ with } r_{-1} = -0.1 \text{ and } r_{-2} = -0.11 \,; \tag{3}$$

for $n = 1, 2, 3, \ \ldots$.

See that the recurrence $r_{-(n+1)} = \frac{11}{10} r_{-n} - \frac{1}{10} r_{-(n-1)}$ has Horadam characteristic equation given by

$$x^2 - \frac{11}{10} x + \frac{1}{10} = 0 \,, \tag{4}$$

whose roots are $x_1 = \dfrac{1}{10}$ and $x_2 = 1$. We find $c_1 = \frac{-1}{9}$ and $c_2 = \frac{1}{9}$. Then, the Binet's formula from the repunit sequence with negative index, as follows.

Proposition 2.1. *[11] For all $n \geq 1$, we have*

$$r_{-n} = -\frac{10^n - 1}{9 \cdot 10^n} \,, \tag{5}$$

where $\{r_{-n}\}_{n\geq 1}$ is the repunit sequence.

3. Tridiagonal repunit matrix

In this section, we will investigate the tridiagonal matrices and their connections to repunit sequences.

We remember that a tridiagonal matrix is a square matrix of order n in which the non-zero elements are located only on the main diagonal, the subdiagonal, and the superdiagonal. In other words, the non-zero elements are those on the main diagonal and those located immediately above and below it.

Let us consider a square matrix $\mathcal{M}_n$ of order $n \geq 1$ defined by:

$$\mathcal{M}_n = \begin{bmatrix} a & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & d & e & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & d & e & \cdots & 0 & 0 & 0 \\ 0 & 0 & c & d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d & e & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & d & e \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & d \end{bmatrix}, \tag{6}$$

where $a,\ b,\ c,\ d$ and e are non-zero real numbers.

Initially, the auxiliary result that we present below.

Lemma 3.1. *[12, 13] The matrix $\mathcal{M}_n$ is tridiagonal and for all $n \geq 2$ we have:*

$$|\mathcal{M}_{n+1}| = d|\mathcal{M}_n| - ce|\mathcal{M}_{n-1}| \,.$$

Here, we consider certain special tridiagonal matrices that enable us to determine the terms of the repunit numbers through the determinant of such matrices. According to Cahill et al [14] and Falcom [12], a tridiagonal matrix associated with a Horadam-type sequence is a square matrix of order n in which the determinant is the general expression of the sequence. Thus, a tridiagonal matrix associated with the repunit sequence $\{r_n\}_{n\geq 0}$ is presented in [13].

For all $n \geq 1$, in the Equation (6), by taking $a = 11,\ b = -1,\ c = -10,\ d = 11$ and $e = -1$, as shown in the following matrix:

$$\mathcal{R}_n = \begin{bmatrix} 11 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -10 & 11 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -10 & 11 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 11 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -10 & 11 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -10 & 11 \end{bmatrix}. \tag{7}$$

The matrix $\mathcal{R}_n$ is the repunit tridiagonal matrix.
Example 3.2. The direct calculation of the determinant of the repunit tridiagonal matrix $\mathcal{R}_n$ for $n = 1$ and $n = 2$ shows the following values

$$|\mathcal{R}_1| = 11 = r_2, \text{ and } |\mathcal{R}_2| = 11 \cdot 11 - (-1) \cdot 10 = 111 = r_3 .$$

We will apply the same procedure for $n = 3$ and $n = 4$.

$$\begin{aligned}
|\mathcal{R}_3| = &\begin{vmatrix} 11 & -1 & 0 \\ -10 & 11 & -1 \\ 0 & -10 & 11 \end{vmatrix} = 11 \begin{vmatrix} 11 & -1 \\ -10 & 11 \end{vmatrix} - (-1) \begin{vmatrix} -10 & -1 \\ 0 & 11 \end{vmatrix} \\
=&11|\mathcal{R}_2| - (-1)(-10)|\mathcal{R}_1| = 1111 = r_4 \\
|\mathcal{R}_4| = &\begin{vmatrix} 11 & -1 & 0 & 0 \\ -10 & 11 & -1 & 0 \\ 0 & -10 & 11 & -1 \\ 0 & 0 & -10 & 11 \end{vmatrix} = 11 \begin{vmatrix} 11 & -1 & 0 \\ -10 & 11 & -1 \\ 0 & -10 & 11 \end{vmatrix} - (-1) \begin{vmatrix} -10 & -1 & 0 \\ 0 & 11 & -1 \\ 0 & -10 & 11 \end{vmatrix} \\
=&11|\mathcal{R}_3| + 11 \begin{vmatrix} -10 & -1 \\ 0 & 11 \end{vmatrix} - (-10) \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix} \\
=&11|\mathcal{R}_3| - (-1)(-10)|\mathcal{R}_2| = r_5 .
\end{aligned}$$

This procedure can be generalized to any $n > 3$, it is shown that in Proposition 3.3. Therefore, note that
Proposition 3.3. *[13] Consider the tridiagonal matrix $\mathcal{R}_n$ of order n given in the Equation* (7). *For all $n \geq 1$, we have that*

$$|\mathcal{R}_n| = r_{n+1} ,$$

where $\{r_n\}_{n\geq 0}$ is the repunit sequence.

Remember that a matrix A is said to be invertible (or non-singular) when there exists another matrix denoted by A^{-1} such that $A^{-1} \cdot A = A \cdot A^{-1} = I$, where I is the identity matrix of order n. Let us see an example.

Example 3.4. Let $A = \begin{bmatrix} 11 & -1 & 0 & 0 \\ -10 & 11 & -1 & 0 \\ 0 & -10 & 11 & -1 \\ 0 & 0 & -10 & 11 \end{bmatrix}$ be a tridiagonal matrix. Then its inverse is given by $A^{-1} = \begin{bmatrix} \frac{1111}{11111} & \frac{111}{11111} & \frac{11}{11111} & \frac{1}{11111} \\ \frac{1110}{11111} & \frac{1221}{11111} & \frac{121}{11111} & \frac{11}{11111} \\ \frac{1100}{11111} & \frac{1210}{11111} & \frac{1221}{11111} & \frac{111}{11111} \\ \frac{1000}{11111} & \frac{1100}{11111} & \frac{1110}{11111} & \frac{1111}{11111} \end{bmatrix}$.

In general, as in Example 3.4, a tridiagonal matrix does not have a tridiagonal matrix inverse. This makes it difficult to establish a direct connection between the matrix $\mathcal{R}_n$ and the negative indices of r_n through the direct inversion of $\mathcal{R}_n$.

Now, we presented a tridiagonal matrix associated with repunit sequence $\{r_{-n}\}_{n\geq 0}$. Similarly, by substituting $a = -\frac{11}{100}$, $b = \frac{1}{10}$, $c = -\frac{1}{10}$, $d = \frac{11}{10}$ and $e = -1$ into the matrix in the Equation (6), we find:

$$\mathcal{R}_{-n} = \begin{bmatrix} -\frac{11}{100} & \frac{1}{10} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{11}{10} & -1 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{10} & \frac{11}{10} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{11}{10} & -1 & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{10} & \frac{11}{10} & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{10} & \frac{11}{10} \end{bmatrix}. \tag{8}$$

The matrix $\mathcal{R}_{-n}$ presented in the Equation (8) is an extension of the repunit tridiagonal matrix at negative indices or simply repunit tridiagonal matrix $\mathcal{R}_{-n}$.

Example 3.5. The direct calculation of the determinant of tridiagonal repunit matrix $\mathcal{R}_{-n}$ for $n = 1$, $n = 2$, and $n = 3$ is

$$\begin{aligned} |\mathcal{R}_{-1}| &= -\frac{11}{100} = r_{-2}\,; \\ |\mathcal{R}_{-2}| &= \begin{vmatrix} -\frac{11}{100} & \frac{1}{10} \\ -\frac{1}{10} & \frac{11}{10} \end{vmatrix} = \frac{-11}{100}\cdot\frac{11}{10} - \frac{1}{10}\cdot\frac{(-1)}{10} = -\frac{111}{1000} = r_{-3}\,; \\ |\mathcal{R}_{-3}| &= \begin{vmatrix} -\frac{11}{100} & \frac{1}{10} & 0 \\ -\frac{1}{10} & \frac{11}{10} & -1 \\ 0 & -\frac{1}{10} & \frac{11}{10} \end{vmatrix} \\ &= \frac{-11}{100}\cdot\frac{11}{10}\cdot\frac{11}{10} + 0 + 0 - \frac{-11}{100}\cdot\frac{-1}{10}\cdot(-1) - \frac{-1}{10}\cdot\frac{1}{10}\cdot\frac{11}{10} - 0 \\ &= -\frac{11^3}{10^4} + \frac{11}{10^3} + \frac{11}{10^3} = \frac{-1331+220}{10^4} = -\frac{1111}{10^4} = r_{-4}\,. \end{aligned}$$

We exhibit in the Example 3.5 that by considering $n = 1$, $n = 2$ or $n = 3$ in the Equation (8), we obtain r_{-2}, r_{-3} or r_{-4}, respectively, from the determinant of the matrix $\mathcal{R}_{-n}$.

The Proposition 3.6 shows that for $n \geq 3$, the determinant of $\mathcal{R}_{-n}$ provides the predecessor term in the repunit sequence with negative subscript. Thus, we have

Proposition 3.6. *Consider the repunit tridiagonal matrix $\mathcal{R}_{-n}$ of order n given in the Equation* (8)*. For all $n \geq 1$, we have $|\mathcal{R}_{-n}| = r_{-(n+1)}$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.*

Proof. For $n = 1$, the result follows from Example 3.5.

On the other hand, for $n \geq 2$, it suffices to apply Lemma 3.1, and we obtain that

$$|\mathcal{R}_{-(n+1)}| = \frac{11}{10}|\mathcal{R}_{-n}| - \left(-\frac{1}{10}\right)\cdot(-1)|\mathcal{R}_{-(n-1)}|\,.$$

Now, using induction on n, assume that the property is true for all values less than or equal to n.

We have $|\mathcal{R}_{-n}| = r_{-(n+1)}$ and $|\mathcal{R}_{-(n-1)}| = r_{-n}$. So, according Equation (3)

$$\begin{aligned} |\mathcal{R}_{-(n+1)}| &= \frac{11}{10} r_{-(n+1)} - \left(\frac{1}{10}\right) \cdot r_{-(n+1)} \\ &= r_{-(n+2)} \,. \end{aligned}$$

Therefore, by the principle of mathematical induction, the property is true for all non-negative integers n. □

4. Second tridiagonal repunit matrix

According [13], consider the other tridiagonal matrix $\mathcal{R}'_n$ of order $n+1$ given by

$$\mathcal{R}'_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -10 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 11 & -10 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 11 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 11 & -10 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 11 \end{bmatrix} . \tag{9}$$

Example 4.1. Note that when we calculate the determinant of $\mathcal{R}'_n$ for $n = 0, 1, 2$, and 3, we obtain that

$$|\mathcal{R}'_0| = 0 = r_0 \text{ and } |\mathcal{R}'_1| = 0 \cdot 0 - (-1) \cdot 1 = 1 = r_1 \,,$$

$$|\mathcal{R}'_2| = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -10 \\ 0 & -1 & 11 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} -1 & -10 \\ 0 & 11 \end{vmatrix} = 11 = r_2,$$

$$\begin{aligned} |\mathcal{R}'_3| &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -10 & 0 \\ 0 & -1 & 11 & -10 \\ 0 & 0 & -1 & 11 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} -1 & -10 & 0 \\ 0 & 11 & -10 \\ 0 & -1 & 11 \end{vmatrix} \\ &= -\left((-1)(-1)^{1+1} \begin{vmatrix} 11 & -10 \\ -1 & 11 \end{vmatrix}\right) = 111 = r_3 \,. \end{aligned}$$

When we analyze the determinant for the cases, we notice that this procedure can be generalized by Proposition 4.2 below.

Proposition 4.2. *[13] Consider the tridiagonal matrix $\mathcal{R}'_n$ of order $n+1$ given in the Equation* (9)*. For all $n \geq 0$, the n-th repunit number is given by $|\mathcal{R}'_n| = r_n$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.*

To prove this result, we simply use the Laplace expansion for the determinant. This is the same expansion we used in Example 4.1.

Similarly, let the matrix $\mathcal{R}'_{-n}$ of order $n+1$ be given by

$$\mathcal{R}'_{-n} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} & 0 & \cdots & 0 & 0 \\ 0 & -1 & \frac{11}{10} & -\frac{1}{10} & \cdots & 0 & 0 \\ 0 & 0 & -1 & \frac{11}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & \cdots & -1 & \frac{11}{10} \end{bmatrix}. \tag{10}$$

We will analyze some particular cases to illustrate the calculation of the determinant.

Example 4.3. For $n=1$, the result is easily obtained, just note that

$$|\mathcal{R}'_{-1}| = 0 \cdot 0 - \frac{1}{10} \cdot 1 = -\frac{1}{10} = r_{-1}\,.$$

Now, for $n=2$ and $n=3$, we have

$$|\mathcal{R}'_{-2}| = \begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} \\ 0 & -1 & \frac{11}{10} \end{vmatrix} = 1 \cdot (-1)^{1+2} \begin{vmatrix} \frac{1}{10} & -\frac{1}{10} \\ 0 & \frac{11}{10} \end{vmatrix} = -\frac{11}{100} = r_{-2}\,,$$

and,

$$\begin{aligned} |\mathcal{R}'_{-3}| &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} & 0 \\ 0 & -1 & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & -1 & \frac{11}{10} \end{vmatrix} = -\begin{vmatrix} \frac{1}{10} & -\frac{1}{10} & 0 \\ 0 & \frac{11}{10} & -\frac{1}{10} \\ 0 & -1 & \frac{11}{10} \end{vmatrix} \\ &= -\frac{1}{10} \begin{vmatrix} \frac{11}{10} & -\frac{1}{10} \\ -1 & \frac{11}{10} \end{vmatrix} = -\frac{111}{1000} = r_{-3}\,. \end{aligned}$$

We now present an auxiliary result that will be useful in the following result.

Lemma 4.4. *Consider the tridiagonal matrix U_n of order n given by*

$$U_n = \begin{bmatrix} \frac{11}{10} & -\frac{1}{10} & 0 & \cdots & 0 & 0 \\ -1 & \frac{11}{10} & -\frac{1}{10} & \cdots & 0 & 0 \\ 0 & -1 & \frac{11}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & \cdots & -1 & \frac{11}{10} \end{bmatrix}.$$

For all $n \geq 1$, then $-\frac{1}{10}|U_n| = r_{-(n+1)}$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.

In a manner similar to Proposition 3.6, the proof of Lemma 4.4 can be carried out using induction on n.

To conclude this section, we highlight an interesting result that connects the determinant of the matrix $\mathcal{R}'_{-n}$ to negative index repunit. Namely, we have:

Proposition 4.5. *Consider the tridiagonal matrix $\mathcal{R}'_n$ of order $n+1$ given in the Equation* (10). *For all $n \geq 1$, then $|\mathcal{R}'_{-n}| = r_{-n}$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.*

Proof. In Example 4.3 displays the determinant of $\mathcal{R}'_n$ up to $n = 3$. For $n \geq 3$ and using Laplace's expansion, we have

$$\begin{aligned}
|\mathcal{R}'_{-n}| &= \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{10} & 0 & -\frac{1}{10} & 0 & \cdots & 0 & 0 \\ 0 & -1 & \frac{11}{10} & -\frac{1}{10} & \cdots & 0 & 0 \\ 0 & 0 & -1 & \frac{11}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & \cdots & -1 & \frac{11}{10} \end{vmatrix} \\
&= 1 \cdot (-1)^{1+2} \begin{vmatrix} \frac{1}{10} & -\frac{1}{10} & 0 & \cdots & 0 & 0 \\ 0 & \frac{11}{10} & -\frac{1}{10} & \cdots & 0 & 0 \\ 0 & -1 & \frac{11}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & \cdots & -1 & \frac{11}{10} \end{vmatrix} \\
&= -\frac{1}{10} \cdot (-1)^{1+1} \begin{vmatrix} \frac{11}{10} & -\frac{1}{10} & \cdots & 0 & 0 \\ -1 & \frac{11}{10} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & \cdots & -1 & \frac{11}{10} \end{vmatrix} \\
&= -\frac{1}{10} |U_{n-1}| \,.
\end{aligned}$$

Applying Lemma 4.4, we conclude that $|\mathcal{R}'_{-n}| = r_{-n}$, and this completes the proof. □

5. Third tridiagonal matrix

In this section, we present a third form of a tridiagonal matrix. In particular, we will apply the determinant of this type of matrix connecting with the repunit sequence, remember, a type of Horadam sequence.

This matrix was defined by [15, 16]. To construct a $n \times n$ tridiagonal matrix $\mathcal{T}_n = [t_{ij}]$ with entries $t_{k,k} = x_1 + x_2$; $t_{k,k+1} = x_2$, and $t_{k+1,k} = x_1$ for $1 \leq k \leq n-1$; that is,

$$\mathcal{T}_n = \begin{bmatrix} x_1+x_2 & x_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & x_1+x_2 & x_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x_1 & x_1+x_2 & x_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_1+x_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_1+x_2 & x_2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x_1 & x_1+x_2 & x_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x_1 & x_1+x_2 \end{bmatrix}, \tag{11}$$

where x_1, x_2 are real or complex numbers such that $x_1x_2 \neq 0$ and $(x_1+x_2)^2 \neq 4x_1x_2$.

Example 5.1. Let us observe that, for $n = 1, 2, 3$, we have:

$$\begin{aligned} |\mathcal{T}_1| &= x_1 + x_2 \,; \\ |\mathcal{T}_2| &= \begin{vmatrix} x_1+x_2 & x_2 \\ x_1 & x_1+x_2 \end{vmatrix} = x_1^2 + x_1x_2 + x_2^2 \,; \\ |\mathcal{T}_3| &= \begin{vmatrix} x_1+x_2 & x_2 & 0 \\ x_1 & x_1+x_2 & x_2 \\ 0 & x_1 & x_1+x_2 \end{vmatrix} \\ &= (x_1+x_2)\begin{vmatrix} x_1+x_2 & x_2 \\ x_1 & x_1+x_2 \end{vmatrix} - x_2 \begin{vmatrix} x_1 & x_2 \\ 0 & x_1+x_2 \end{vmatrix} \\ &= (x_1+x_2)|\mathcal{T}2| - x_1x_2|\mathcal{T}1| = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 \,. \end{aligned}$$

In general, we have that

Proposition 5.2. *[15, 16] Let's $\mathcal{T}_n$ the tridiagonal matrix of order n defined by the Equation* (11). *Then, for all $n \geq 1$*

$$|\mathcal{T}_n| = \sum_{k=0}^{n} x_1^{n-k} x_2^k \,.$$

According [15, 16], by rewriting Proposition 5.2 as described above, we arrive at the following result

$$\begin{aligned} |\mathcal{T}_n| = &\sum_{k=0}^{n} x_1^{n-k} x_2^k \;= x_1^n + x_1^{n-1}x_2 + x_1^{n-2}x_2^2 + \ldots + x_1x_2^{n-1} + x_2^n \\ = &\frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} \,. \end{aligned} \tag{12}$$

Let x_1 and x_2 be the distinct roots of the equation $x^2 - px + q = 0$, the Horadam characteristic equation of recurrence.

Specifying $x_1 = 10$ and $x_2 = 1$ into Proposition 5.2, it coincides with the Binet Formula for repunit numbers, Equation (2), thus showing that

Proposition 5.3. *Let's $\mathcal{T}_n$ the tridiagonal matrix of order n defined by the Equation* (11)*, with $x_1 = 10$ and $x_2 = 1$. Then, for all $n \geq 1$, $|\mathcal{T}_n| = r_{n+1}$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.*

Similarly, substituting $x_1 = \frac{1}{10}$ and $x_2 = 1$ in Proposition 5.2, it is shown that

Proposition 5.4. *Let's $\mathcal{T}_n$ the tridiagonal matrix of order n defined by the Equation* (11)*, with $x_1 = \frac{1}{10}$ and $x_2 = 1$. Then, for all $n \geq 1$, $|\mathcal{T}_n| = r_{-(n+1)}$, where $\{r_n\}_{n\geq 0}$ is the repunit sequence.*

6. Final Considerations

Sequences exhibiting specific periodicity properties, notably the repunit, find application in digital representations and theoretical computer science, thus rendering them pertinent to discrete mathematics and cryptography. Tridiagonal matrices, defined by nonzero elements exclusively on the main diagonal and the diagonals directly above and below it, find numerous applications in diverse fields of science, mathematics, and engineering.

In this work, we extended the tridiagonal matrices presented in [13] for the repunit sequence $\{r_n\}_{n\geq 0}$ to include the repunit sequence with negative indices $\{r_{-n}\}_{n\geq 1}$. Additionally, we presented a direct application of a result from [15, 16], which linked tridiagonal matrices associated with second-order linear recurrences to the repunit sequence for all integers n. The paper presented a novel application of tridiagonal matrices to the repunit sequence, an area that has been less explored in mathematical literature. The elegance of the results lay in the simplicity of their proofs, which were neither overly complex nor lengthy. They primarily relied on the use of Lagrange identities for calculating determinants, followed by induction to establish the results.

Through this approach, we aimed to inspire further studies on this class of numbers and the tridiagonal matrices, potentially leading to the discovery of new results. The exploration of matrix-based representations of these sequences could inspire other studies in the combinatorial theory of matrices.

7. Acknowledgments

The first author was partially supported by PROPESQ-UFT. The last author is member of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT—Centre of Mathematics of University of Minho) and she thanks the Portuguese Funds through FCT—Fundação para a Ciência e a Tecnologia, within the Projects UIDB/00013/2020 and UIDP/00013/2020..

The authors are grateful to the reviewers for the suggestions to improve the paper.

References

[1] Sloane NJA, et al.. The on-line encyclopedia of integer sequences; 2003. Available from: https://oeis.org/.

[2] Santos DC, Costa EA. A note on Repunit number sequence. Intermaths. 2024;5(1):54-66.

[3] Horadam AF. A generalized Fibonacci sequence. The American Mathematical Monthly. 1961;68(5):455-9.

[4] Horadam AF. Basic properties of a certain generalized sequence of numbers. The Fibonacci Quarterly. 1965;3(3):161-76.

[5] Cerda G. Matrix methods in Horadam sequences. Boletíns de Matemáticas. 2012;19(2):97-106.

[6] Jaroma JH. Factoring Generalized Repunits. Bulletin of the Irish Mathematical Society. 2007;59:29-35.

[7] Kalman D, Mena R. The Fibonacci numbers-exposed. Mathematics magazine. 2003;76(3):167-81.

[8] Beiler AH. Recreations in the Theory of Numbers: The Queen of Mathematics Entertains. 2nd ed. Cambridge MA: Dover; 1966.

[9] Costa EA, Santos DC. Algumas propriedades dos números monodígitos e das repunidades. Revista de Matemática da UFOP. 2022;2(1):47-58.

[10] Yates S. Repunits and repetends. Star Publishing Co., Inc; 1982.

[11] Costa EA, Santos DC, Monteiro FS, Souza VMA. On the repunit sequence at negative indices. Revista de Matemática da UFOP. 2024;1(1):1-12.

[12] Falcon S. On the generating matrices of the k-Fibonacci numbers. Proyecciones. 2013;32(4):347-57.

[13] Santos DC, Costa EA. Números repunidades e a representação matricial. Revista Sergipana de Matemática e Educação Matemática. 2024;9(1):81-96.

[14] Cahill ND, D'enrico JR, Narayan DA, Narayan JY. Fibonacci Determinants. The College Mathematics Journal. 2002;33(3):221-5.

[15] Kilic E, Taşci D. On sums of second order linear recurrences by Hessenberg matrices. The Rocky Mountain Journal of Mathematics. 2008:531-44.

[16] Kilic E, Taşci D. On the second order linear recurrences by tridiagonal matrices. Ars Combin. 2009;91:11-8.