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# Implicit equations Involving The p-Biharmonic Operator*

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***Abstract.*** *In this research, we will study the existence of weak solutions for a class of implicit elliptic equations involving the $p$-biharmonic operator. Using a Krasnoselskii-Schaefer type theorem we establish our result, extending and complementing those obtained by R. Precup, 2020, and P.C. Carrião et al., 2009.*
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## 1. Introduction

In this research we focus on the following boundary elliptic problem:

$$\begin{aligned} \triangle_p^2 u &= f(x,u,\triangle u,\triangle_p^2 u) + g(x,u,\triangle u) \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma$, $(n \geq 3)$, $\triangle_p^2 u = \triangle(|\triangle u|^{p-2}\triangle u)$ is the $p$-biharmonic operator, $2 < p < +\infty$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions.

In 1958, Krasnoselskii proved his famous result on the existence of fixed point for a sum of two operators, one of which is a contraction and the second one is compact, defined in a convex and closed set, and concluding that its sum has a fixed point. Since then, many extensions have emerged with various types of generalized contractions and generalized compact operators, which are generally applied to the resolution of specific problems posed in natural sciences and physics. In particular, his result gives a method for solving Dirichlet problems in which nonlinear sources can be expressed by the sum of two terms to which appropriate restrictions are imposed to fulfill the hypotheses in Krasnoselskii's theorem. Precup [1] studied the Dirichlet problem with the Laplacian operator $(p = 2)$ for implicit equations involving two sources, one source containing the Laplacian

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and another containing the gradient, via a Krasnoselskii-type fixed point theorem and suggested the application of his technique to general elliptic operators that replace the Laplacian and to other classes of implicit differential equations, e.g. equations of type $F(x, u(x), Du(x), D^2u(x)) = 0$, which it is not possible to write it as an equivalent equation linear with respect to the operator with the second derivatives $D^2u$). Let us emphasize that, in this case, equation (1) represents a quasilinear PDE of fourth order with a linear operator, the function $f$ is Lipchiz of linear growth and the function $g$ satisfies a condition of negative sign in the second variable with more flexible growth, but more restrictive in the third variable (see hypothesis $(A_2)$ below). However when $p > 2$, the operator has a strongly nonlinear behavior, the growth of $f$ is sublinear and $g$ has sub $p$-quadratic growth which implies a greater difficulty in the analysis.

Thus, inspired by the ideas introduced by Precup, this paper aims to study the existence of solutions for the implicit equation (1) involving the $p$-biharmonic operator with $p > 2$. This extension is not trivial due to the mathematical difficulties posed by the degenerate quasilinear elliptic operator, compared to the bi-harmonic operator (also known as the bilaplacian): the lack of Hilbert structure of the domain of the operator, the absence of linearity and the complicated spectral properties. Problems like (1) arise in the theory of bending extensible elastic beams on nonlinear elastic foundations; the solution $u = u(x)$ represents a thin extensible elastic beam, while the functions $f$ and $g$ act as a forces exerted on the beam by the foundation. Thus, the problem represents the bending equilibrium of the system. It also allow furnishes a model to study traveling waves in suspension bridges (see [2, 3, 4]). We point out that implicit elliptic equations have been intensively studied in the literature (see [5, 6, 7, 8] and references therein), and have multiple applications to the calculus of variations, nonlinear elasticity, problems of phase transitions and optimal design (see, e.g., [9, 10, 11, 12, 13, 14]).

To our knowledge there are few works related to implicit equations with the $p$-biharmonic operator. In [15], implicit equations containing the $p$-Laplacian operator were investigated using a selection theorem for decomposable-valued multifunctions. In [8], an implicit equation involving a Laplacian-like operator, but looking for solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, was studied in the context of set-valued analysis. We apply the fixed point technique to solve problem (1) which, in our opinion, is a more direct and accessible procedure to solve equations of this type.

It is worth mentioning that in [16] the authors have introduced $p$-polyharmonic operators and in [17] the results was extended to nonlocal higher-order problems. Both situations have been treated via variational methods. In this regard, we suggest analyzing the remark at the end of this work.

The paper is organized as follows. In Section 2, as preliminaries, we recall some properties of the inverse operator of $p$-Laplacian and the main tool, a hybrid theorem of Krasnoselskii type due to Gao et al.[18]. Section 3 is devoted to state and prove our main result about existence of weak solutions for problem (1).

## 2. Preliminaries

Let $W_0^{2,p}(\Omega)$, $(1 < p)$, be the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\triangle u|^p\, dx\right)^{1/p}, \quad u \in W_0^{2,p}(\Omega)$$

and $\|u\|_p = \left(\int_\Omega |u|^p\, dx\right)^{1/p}$ denotes the norm in $L^p(\Omega)$.

By the Sobolev embedding theorem, for any $2 \le \theta \le p^{**} = \frac{Np}{N-2p}$ $(2 \le \theta < p^{**})$, $\quad p < \frac{N}{2}$, the embedding $W_0^{2,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ is continuous (compact) and there exists a positive constant $C_\theta$ such that $\|u\|_\theta \le C_\theta \|u\|$ , $\quad$ for all $u \in W_0^{2,p}(\Omega)$ (See [19]).

Consider the first eigenvalue $\lambda_1$ of the problem

$$\begin{cases} \triangle(|\triangle u|^{p-2}\triangle u) = \lambda_1 |u|^{p-2}u & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

Thanks to the work of Khalil et al. [20], one has that

$$\lambda_1 := \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\triangle u|^p\, dx}{\int_\Omega |u|^p\, dx}$$

is isolated and simple, also its corresponding first eigenfunction is positive. Thus, the best(smallest) embedding constant for the inclusion $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$ is $1/\sqrt[p]{\lambda_1}$ (See [21])

Let $W^{-2,p'}(\Omega)$ be the dual space of $W_0^{2,p}(\Omega)$. Also, an embedding constant for the inclusion $L^{p'}(\Omega) \hookrightarrow W^{-2,p'}(\Omega)$ is $1/\sqrt[p]{\lambda_1}$.

**Remark 2.1.** The embedding constant for this inclusion can be evaluated through $\lambda_1$. In fact

$$\|v\|_{W^{-2,p'}(\Omega)} = \sup_{\|u\|\le 1} \int_\Omega u(x)v(x)\, dx \le \sup_{\|u\|\le 1} \|u\|_p \|v\|_{p'} \le \lambda^{-1/p}\|v\|_{p'}$$

for all $v \in L^{p'}(\Omega)$.

It is well known, that the problem

$$\begin{cases} \triangle_p^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases}$$

has a unique weak solution $u \in W_0^{2,p}(\Omega)$ for $f \in W^{-2,p'}(\Omega)$. Thus, $S = -\triangle_p^2 : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$ has the following properties:

(i) $S$ is bijective and uniformly continuous on bounded sets.

(ii) The operator $S^{-1}: W^{-2,p'}(\Omega) \to W_0^{2,p}(\Omega)$ is continuous and the following estimate holds

$$\|S^{-1}v_1 - S^{-1}v_2\| \leq M_1^{1/(p-1)} \|v_1 - v_2\|_{-2}^{1/(p-1)} \text{ for any } v_1, v_2 \in W^{-2,p'}(\Omega), \quad (2)$$

for some constant $M_1 > 0$ independent of $v_1$ and $v_2$.

(iii)

$$\|Su\|_{-2} = \|u\|^{p-1}, \quad u \in W_0^{2,p}(\Omega), \quad (3)$$

where $\|.\|_{-2}$ denotes the norm in $W^{-2,p'}(\Omega)$, $1/p + 1/p' = 1$.

We recall that our approach is based on a extension of Krasnoselskii's theorem, which combine Banach's contraction principle with Schaefer's fixed point theorem due to Gao, Li and Zhang [18], and on the previous mentioned work [1] by Precup.

**Theorem 2.2** (Gao-Li-Zhang)**.** *Let $D_R$ be a closed ball centered at the origin and of radius $R$ of a Banach space $X$, and let $A, B$ be operators such that*

*(i) $A : X \to X$ is a contraction;*

*(ii) $B : D_R \to X$ is continuous with $B(D_R)$ relatively compact;*

*(iii) $x \neq A(x) + \lambda B(x)$, for all $x \in \partial D_R$ and $\lambda \in ]0, 1[$.*

*Then the operator $A + B$ has at least one fixed point, i.e., there exits $x \in D_R$ such that*

$$x = A(x) + B(x).$$

**Remark 2.3.** In practice, we use the method of a priori estimates, so both operators $A, B$ are defined on the whole space $X$, and a ball $D_R$ as required by condition $(iii)$ of Theorem 2.2 exists if the set

$$Y = \{x \in X | x = A(x) + \lambda B(x), \quad \text{for some } \lambda \in [0, 1]\}$$

is bounded in $X$.

**Lemma 2.4.** *For any $(p^{**})' \leq \tau \leq p$, the embeddings*

$$W_0^{2,p}(\Omega) \hookrightarrow L^\tau(\Omega),\ L^p(\Omega) \hookrightarrow L^\tau(\Omega),\ L^\tau(\Omega) \hookrightarrow W^{-2,p'}(\Omega),$$

*are continuous, and we may consider positive constants $c_1, c_2, c_3$ such that*

$$\|u\|_\tau \leq c_1 \|u\|, \quad \|u\|_\tau \leq c_2 \|u\|_p, \quad \|u\|_{-2} \leq c_3 \|u\|_\tau, \quad (4)$$

*with*

$$c_2 = c_1 \sqrt[p]{\lambda_1}, \quad c_3 = \frac{c_\Omega}{c_1 \lambda_1^{2/p}}, \quad \textit{and} \quad c_\Omega = |\Omega|^{(p-2)/p}.$$

*Proof.* From (4), we get

$$\|u\|_\tau \leq c_2 \|u\|_p \leq \frac{c_2}{\sqrt[p]{\lambda_1}} \|u\|, \quad \text{for } u \in W_0^{2,p}(\Omega),$$

which give us $c_2 = c_1 \sqrt[p]{\lambda_1}$.

On the other hand, in view of the inclusions $L^{p'}(\Omega) \hookrightarrow W^{-2,p'}(\Omega)$, $L^p(\Omega) \hookrightarrow L^{p'}(\Omega)$, we have for $u \in W_0^{2,p}(\Omega)$

$$\|u\|_{-2} \leq \frac{1}{\sqrt[p]{\lambda_1}} \|u\|_{p'} \leq \frac{c_\Omega}{\sqrt[p]{\lambda_1}} \|u\|_p \leq \frac{c_\Omega}{\lambda_1^{2/p}} \|u\|.$$

Now, since

$$\|u\|_{-2} \leq c_3 \|u\|_\tau \leq c_3 c_1 \|u\|,$$

it follows that $c_1 c_3 = c_\Omega / \lambda_1^{2/p}$. □

We seek weak solutions of our problem, i.e. functions $u \in W_0^{2,p}(\Omega)$ with $f(.,u,\triangle u,\triangle_p^2 u) + g(.,u,\triangle u) \in W^{-2,p'}(\Omega)$ and

$$\langle \triangle_p^2 u, v\rangle = \langle f(x,u,\triangle u,\triangle_p^2 u) + g(x,u,\triangle u), v\rangle \quad \text{for all } v \in W_0^{2,p}(\Omega),$$

where $\langle u, v\rangle$ denotes the duality pairing between $W^{-2,p'}(\Omega)$ and $W_0^{2,p}(\Omega)$. Setting $v = Su$, equation (1) is equivalent to the fixed point equation

$$v = f\left(x, S^{-1}v, \triangle S^{-1}v, -v\right) + g\left(x, S^{-1}v, \triangle S^{-1}v\right), \tag{5}$$

which will be solved in the Lebesgue space $L^\tau(\Omega)$ with $\tau \geq (p^{**})'$.

Define operators $A, B : L^\tau(\Omega) \to L^\tau(\Omega)$ by

$$Av = f\left(\cdot, S^{-1}v, \triangle S^{-1}v, -v\right),$$
$$Bv = g\left(\cdot, S^{-1}v, \triangle S^{-1}v\right).$$

Then equation (5) becomes the operator equation

$$v = A(v) + B(v).$$

Our idea is to use Theorem 2.2 to find the fixed point for the sum $A + B$ in $W_0^{2,p}(\Omega)$. For this goal, we need to impose additional conditions on $f$ and $g$ to guarantee that the two operators are well defined from $L^\tau(\Omega)$ to itself, and then, we will show that $A$ is a contraction, and $B$ is completely continuous. We conclude, by establishing a priori bounds for the solutions to the problem as required by Remark 2.3.

## 3. Existence of Solutions

In this section, we present our main result. More precisely, under suitable conditions, we prove the existence of a solution to problem (1) by applying Theorem 2.2.

First, we give the following hypotheses on $f$ and $g$.

$(A_1)$ There exist $a, b, c \geq 0$ such that

$$|f(x,y,z,w) - f(x,\overline{y},\overline{z},\overline{w})| \leq a|y-\overline{y}|^{p-1} + b|z-\overline{z}|^{p-1} + c|w-\overline{w}|,$$
$$f(\cdot\,,0,0,0) \in L^p(\Omega).$$

$(A_2)$ There exist constants $a_0, b_0 \geq 0$, $\alpha \in [1, p^{**}/(p^{**})']$, $\beta \in [1, p/(p^{**})']$, and $h \in L^p(\Omega)$ such that

$$|g(x,y,z)| \leq a_0|y|^\alpha + b_0|z|^\beta + h(x) \quad \text{for any } y \in \mathbb{R}, z \in \mathbb{R}^n \text{ and } a.e\ x \in \Omega.$$

$(A_3)$ $yg(x,y,z) \leq \sigma|y|^p \quad \forall y \in \mathbb{R}, z \in \mathbb{R}^n,\ a.e\ x \in \Omega$, for some $\sigma < \sigma_0\lambda_1,\ 0 < \sigma_0 < 1,\ where\ \lambda_1$ is the first eigenvalue of $(-\triangle_p^2, W_0^{2,p}(\Omega))$ .

$(A_4)$ $\ell_0 := \left(\frac{a}{\lambda_1^{2/p}} + \frac{b}{\lambda_1^{(3-p)/p}}\right)\left(c_1|\Omega|^{\frac{1}{p}-\frac{1}{\tau}}\right)^{p-2} M_1 + c, \quad \ell_1 := \frac{a}{\lambda_1} + \frac{bc_\Omega}{\lambda_1^{(1)/p}} + c,$
$\ell = \max\{\ell_0, \ell_1\} < 1, \quad \sigma_0 = 1 - \ell.$

We are now ready to state our main result.

**Theorem 3.1.** *Let $(p^{**})' \leq \tau \leq p$. Assume that assumptions $(A_1)$–$(A_4)$ hold true. Then (1) has at least one solution $u \in W_0^{2,p}(\Omega)$ with $\triangle_p^2 u \in L^\tau(\Omega)$.*

For the proof of this theorem, we need to establish the following three lemmas.

**Lemma 3.2.** *Suppose that $(A_1)$ and $(A_4)$ hold. Then $A$ is a contraction on $L^\tau(\Omega)$ , $\tau \in [1, p/(p-1)]$, provided $a, b$ are sufficiently small.*

*Proof.* The Carathéodory conditions ensure that for every measurable function $v \in L^\tau(\Omega)$, the function $f(\cdot, S^{-1}v, \triangle S^{-1}v, -v)$ is also measurable. Furthermore

$$\left\|f\left(\cdot, S^{-1}v, \triangle S^{-1}v, -v\right)\right\|_\tau = \left\|f\left(\cdot, S^{-1}v, \triangle S^{-1}v, -v\right) - f(\cdot, 0,0,0)\right\|_\tau$$
$$\leq a\left\||S^{-1}v|^{p-1}\right\|_\tau + b\left\||\triangle S^{-1}v|^{p-1}\right\|_\tau + c\|v\|_\tau \tag{6}$$

But, using the inequalities

$$\|z\|_{\tau(p-1)} \leq c_\tau\|z\|, \quad \text{for all } z \in W_0^{2,p}(\Omega), \tag{7}$$

and

$$\|z\|_{\tau(p-1)} \leq c_p\|z\|_p, \quad \text{for all } z \in L^p(\Omega), \tag{8}$$

where $c_\tau$ and $c_p$ are the best constants for the embeddings $W_0^{2,p}(\Omega) \hookrightarrow L^{\tau(p-1)}(\Omega)$ and $L^p(\Omega) \hookrightarrow L^{\tau(p-1)}(\Omega)$ respectively, we have for $v \in L^\tau(\Omega)$

$$\left\||S^{-1}v|^{p-1}\right\|_\tau \leq \left\||S^{-1}v|\right\|_{\tau(p-1)}^{p-1} \leq c_\tau^{p-1}\|S^{-1}v\|^{p-1} = c_\tau^{p-1}\|v\|_{-2}$$
$$\leq c_\tau^{p-1}c_3\|v\|_\tau < \infty$$

and, similarly

$$\left\||\triangle S^{-1}v|^{p-1}\right\|_\tau \leq c_p^{p-1}c_3\|v\|_\tau < \infty.$$

So, from (6) and the above inequalities, $A$ is well defined from $L^{\tau}(\Omega)$ to itself.

Furthermore, we can use this last process to obtain

$$\begin{aligned}
\|Av - Aw\|_{\tau} &\leq a \left\|S^{-1}v - S^{-1}w\right\|_{\tau(p-1)}^{p-1} + b \left\|\triangle S^{-1}v - \triangle S^{-1}w\right\|_{\tau(p-1)}^{p-1} \\
&\quad + c\|v - w\|_{\tau} \\
&\leq a c_{\tau}^{p-1} M_1 c_3 \|v - w\|_{\tau} + b c_p^{p-1} M_1 c_3 \|v - w\|_{\tau} + c\|v - w\|_{\tau} \\
&\leq \left[(a c_{\tau}^{p-1} + b c_p^{p-1}) M_1 c_3 + c\right] \|v - w\|_{\tau} \\
&= \left[\left(\frac{a}{\lambda_1^{2/p}} + \frac{b}{\lambda_1^{(3-p)/p}}\right) \left(c_1 |\Omega|^{1/p - 1/\tau}\right)^{p-2} M_1 + c\right] \|v - w\|_{\tau}.
\end{aligned}$$

It follows, from hypothesis $(A_4)$, that $A$ is a contraction. □

**Lemma 3.3.** *Suppose that $(A_2)$ is satisfied. Then the operator $B : L^{\tau}(\Omega) \longrightarrow L^{\tau}(\Omega)$ is well-defined and completely continuous for*

$$\tau = \min\left\{p^{**}/\alpha, p/\beta\right\}. \tag{9}$$

*Proof.* It is easily checked that (9) implies $(p^{**})' < \tau \leq p$.

We define three operators

$$\begin{aligned}
&I_2 : L^{\tau}(\Omega) \longrightarrow W^{-2,p'}(\Omega), \quad I_2(v) = v, \\
&I_1 : W^{-2,p'}(\Omega) \longrightarrow L^{p^*}(\Omega) \times L^p(\Omega, \mathbb{R}^n), \quad I_1(v) = \left(S^{-1}v, \triangle S^{-1}v\right), \\
&\Phi : L^{p^*}(\Omega) \times L^p(\Omega; \mathbb{R}^n) \longrightarrow L^{\tau}(\Omega), \quad \Phi(u, v) = g(\cdot, u, v).
\end{aligned}$$

We observe that

(i) $I_2$ is completely continuous, since $L^{\tau}(\Omega) \hookrightarrow W^{-2,p'}(\Omega)$ is compact.

(ii) $I_1$ is continuous and bounded, because $\|u\|_{-2} \leq c_3 \|u\|_{\tau}$.

(iii) $\Phi$ is continuous and bounded for $\tau = \min\left\{\frac{p^{**}}{\alpha}, \frac{p}{\beta}\right\}$.
Indeed

$$\begin{aligned}
\|\Phi(u, v)\|_{\tau}^{\tau} &\leq \int_{\Omega} \left(3 \max\left\{a_0 |u|^{\alpha}, b_0 |v|^{\beta}, |h|\right\}\right)^{\tau} dx \\
&\leq 3^{\tau} \left(a_0^{\tau} \|u\|_{\alpha\tau}^{\alpha} + b_0^{\tau} \|v\|_{\beta\tau}^{\beta} + \|h\|_{\tau}^{\tau}\right) \\
&\leq c\left(\|u\|_{p^*}^{\alpha} + \|v\|_p^{\beta} + \|h\|_{\tau}^{\tau}\right).
\end{aligned}$$

Since $g$ is a Carathéodory function, by using Lebesgue's dominated convergence theorem, we obtain the continuity of $\Phi$.

Thus, $B = \Phi \circ I_1 \circ I_2 : L^{\tau}(\Omega) \longrightarrow L^{\tau}(\Omega)$ is a completely continuous operator. □

**Lemma 3.4.** *Suppose that the hypotheses of Lemmas 3.2 and 3.3 are satisfied and, in addition, $g$ satisfies $(A3)$. Then the set*

$$F = \{v \in L^{\tau}(\Omega) : v = Av + \lambda Bv, \quad \text{for some } \lambda \in ]0,1[\}$$

*is bounded in $L^{\tau}(\Omega)$.*

*Proof.* First, we will verify that the set of the solutions is bounded in $W^{-2,p'}(\Omega)$. Let $v \in F$ . By Lemma 2.4, $v \in W^{-2,p'}(\Omega)$, and we have

$$\langle v, S^{-1}v\rangle = \langle Av, S^{-1}v\rangle + \lambda\langle Bv, S^{-1}v\rangle. \tag{10}$$

Now, by the properties of the operator $S^{-1}$, we have $\langle v, S^{-1}v\rangle = \|v\|_{-2}^{p/(p-1)}$, and hence, using $(A_1)$ and $(A_3)$, we can write

$$\begin{aligned}
\|v\|_{-1}^{p/(p-1)} =& \int_{\Omega} f\left(x, S^{-1}v, \triangle S^{-1}v, -v\right) S^{-1}v\,dx \\
&+ \int_{\Omega} g\left(x, S^{-1}v, \triangle S^{-1}v\right) S^{-1}v\,dx \\
&\leq \int_{\Omega} \left(a|S^{-1}v|^{p-1} + b|\triangle S^{-1}v|^{p-1} + c|v| + |f(x,0,0,0)|\right)|S^{-1}v|\,dx \\
&+\sigma\int_{\Omega}|S^{-1}v|^p\,dx \\
&\leq a\|S^{-1}v\|_p^p + b\|\triangle S^{-1}v\|_p^{p-1}\|S^{-1}v\|_{p'} + \|\gamma_0\|_{p'}\|S^{-1}(v)\|_p \\
&+c\int_{\Omega}|v||S^{-1}v|\,dx + \sigma\|S^{-1}v\|_p^p \\
&\leq \left(\frac{a}{\lambda_1} + \frac{bc_{\Omega}}{\lambda_1^{1/p}} + \frac{\sigma}{\lambda_1}\right)\|S^{-1}v\|^p + c\|v\|_{-2}\|S^{-1}v\| + \frac{\|\gamma_0\|_p c_{\Omega}}{\lambda_1^{1/p}}\|S^{-1}v\| \\
&\leq \left(\frac{a}{\lambda_1} + \frac{bc_{\Omega}}{\lambda_1^{1/p}} + c + \frac{\sigma}{\lambda_1}\right)\|v\|_{-2}^{p/(p-1)} + \frac{\|\gamma_0\|_p c_{\Omega}}{\lambda_1^{1/p}}\|v\|_{-2}^{1/(p-1)},
\end{aligned}$$

where $\gamma_0(x) = |f(x,0,0,0)|$. Then, from hypothesis $(A_4)$,

$$\|v\|_{-2}^{p/(p-1)} \leq \frac{\|\gamma_0\|_p c_{\Omega}}{\lambda_1^{1/p}}\|v\|_{-2}^{1/(p-1)}.$$

Therefore,

$$\|v\|_{-2} \leq K_1,$$

where $K_1 = \|\gamma_0\|_p c_{\Omega}/\lambda_1^{1/p}$.

Finally, we will prove that $\|v\|_{\tau} \leq K$ for all $v \in F$ and $K > 0$.

As $\alpha\tau \leq p^{**}$ and $\beta\tau \leq p$, we get

$$\begin{aligned}\|B(v)\|_\tau^\tau = \|\Phi(S^{-1}v, \triangle S^{-1}v)\|_\tau^\tau \leq & C_0 \left( \|S^{-1}v\|_{\alpha\tau}^\alpha + \|\triangle S^{-1}v\|_{\beta\tau}^\beta + \|h\|_\tau^\tau \right) \\ \leq & \tilde{C}_0 \left( \|S^{-1}v\|^\alpha + \| S^{-1}v\|^\beta + \|h\|^\tau \right) \\ = & \tilde{C}_0 \left( \|v\|_{-2}^{\alpha/(p-1)} + \| v\|_{-2}^{\beta/(p-1)} + \|h\|_{-2}^{\tau/(p-1)} \right).\end{aligned}$$

Hence, for $v \in F$, we have

$$\|v\|_\tau \leq \|A(v)\|_\tau + \lambda \|B(v)\|_\tau \leq l\|v\|_\tau + \gamma + K_2,$$

where $\gamma = \|f(., 0, 0, 0)\|_\tau$. This implies $\|v\|_\tau \leq K_2 + \gamma/(1 - l)$, and the proof of this lemma is complete. □

*Proof of Theorem 3.1.* It follows at once from Lemmas 3.2-3.4 and Theorem 2.2. □

Finally, we would like to point out that the existence result for the elliptic problem

$$u \in H^2 \cap H_0^1(\Omega), \quad \triangle^2 u + q\triangle u + \alpha(x)u = f(x, u, \nabla u, \triangle u),$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, obtained by P.C. Carrião et al. [22], can be approached with the techniques presented in this paper.

**Remark 3.5.** It seems to be interesting to study a similar result for the implicit $p$-Kirchoff type problem

$$-M\left(\int_\Omega |\triangle u|^p\, dx\right) \triangle_p^2 u = f(x, u, \triangle u, \triangle_p^2 u) + g(x, u, \triangle u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma,$$

where $M : [0, +\infty) \to [m_0, +\infty)$, $m_0 > 0$ is a continuous function and the same hypotheses of this paper on $f$ and $g$.

We plan to address these questions in a future research.

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