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Jacobsthal-Mulatu Numbers*

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Abstract. *The aim of this study is to find a sequence of Jacobsthal-type numbers, which we will denote by $\{JM_n\}_{n\geq 0}$ and name as the sequence of Jacobsthal-Mulatu, such that each term of the Jacobsthal-Lucas sequence, denoted by $\{j_n\}_{n\geq 0}$, is an average term between JM_n and J_n, where $\{J_n\}_{n\geq 0}$ is the classical Jacobsthal sequence. We investigated some key characteristics of the sequence of Jacobsthal-Mulatu. In this study, we present some interesting properties of these sequences of numbers explored in connection with the sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$.*
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1. Introduction

The Fibonacci sequence, $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$, is known such that each term of this sequence is the sum of the previous two and is defined by the recurrence relation $F_n = F_{n-1}+F_{n-2}$ such that $n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence is listed as $A000045$ in the OEIS [1]. Moreover, Lucas sequence, whose recurrence relation is

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$L_n = L_{n-1} + L_{n-2}$ such that $n \geq 2$ and initial values are $L_0 = 2$ and $L_1 = 1$, is known as sequence $A000032$ in the OEIS [1]. It is one of the well-known mathematical facts and has a considerable number of applications, see [2, 3, 4]. Recently, in [5], by changing the initial terms, the Fibonacci-Mulatu sequence is defined by $FM_n = FM_{n-1} + FM_{n-2}$ such that $n \geq 2$ with the initial values $FM_0 = 4$ and $FM_1 = 1$, and this sequence is cataloged as $A022095$ in the OEIS [1]. We can cite here recent papers [5, 6, 7, 8], where Fibonacci-Mulatu numbers are considered.

The Jacobsthal sequence, $\{0, 1, 1, 3, 5, 11, 21, \ldots\}$, is the sequence in which the terms are defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ such that $n \geq 2$ with the initial values $J_0 = 0$ and $J_1 = 1$. Analogously to the Lucas sequence, the Jacobsthal–Lucas sequence [9] is defined by the same recurrence relation $j_n = j_{n-1} + 2j_{n-2}$ such that $n \geq 2$ and initial values $j_0 = 2$ and $j_1 = 1$.

Theorem 5 from [7] established that the Lucas sequence is the arithmetic mean of the Fibonacci and Fibonacci–Mulatu sequences, that is,

$$2L_n = F_n + FM_n\,, \qquad \text{for all } n \in \mathbb{N}\,. \tag{1}$$

Equation (1) determines when a sequence has the suffix Mulatu. Consider three distinct sequences $A = \{a_n\}_{n\geq 0}$, $B = \{b_n\}_{n\geq 0}$ and $C = \{c_n\}_{n\geq 0}$ with the same recurrence relation, where B is the Lucas type of A. If

$$2b_n = a_n + c_n\,, \qquad \text{for all } n \in \mathbb{N},$$

then we say that the sequence C has the suffix Mulatu, hence, C is an A-Mulatu sequence. In this case, for Equation (1), $a_n = F_n$, $b_n = L_n$, and $c_n = FM_n$.

This study aims to systematize a new sequence of the Jacobsthal type. By changing the initial terms, we will define the *Jacobsthal–Mulatu* sequence $\{JM_n\}_{n\geq 0}$, defined by the recurrence relation

$$JM_n = JM_{n-1} + 2JM_{n-2}\,, \quad n \geq 2, \tag{2}$$

with the initial values $JM_0 = 4$ and $JM_1 = 1$. Table 1 presents several terms of Jacobsthal, Jacobsthal–Lucas, and Jacobsthal–Mulatu numbers, and the respective identification in OEIS [1].

Table 1. First few terms of Jacobsthal, Jacobsthal–Lucas, Jacobsthal–Mulatu sequences.

n	0	1	2	3	4	5	6	7	8	9	10	id OEIS
J_n	0	1	1	3	5	11	21	43	85	171	341	$A001045$
j_n	2	1	5	7	17	31	65	127	257	511	1025	$A014551$
JM_n	4	1	9	11	29	51	109	211	429	851	1709	$A344109$

Let $\{h_n\}$ be the *Horadam sequence* defined for all natural numbers n by the second-order linear recurrence relation $h_{n+1} = -ph_n + qh_{n-1}, \quad n \geq 1$, where p and q are fixed integers, and the initial conditions are given by $h_0 = a$ and $h_1 = b$. This sequence was introduced by Horadam [10, 11, 12] and generalizes several well-known sequences whose recurrence relations correspond to the characteristic equation $x^2 + px - q = 0$. The Horadam sequence is specified in the Jacobsthal sequence if we let $p = -1; q = 2; a = 0; b = 1$. Further general results and extensions concerning the Horadam sequence can be found in [10, 13].

Our research problem is the determination of properties for the Jacobsthal–Mulatu sequence analogous to the Jacobsthal sequence and relationships between three sequences: Jacobsthal, Jacobsthal–Lucas, and Jacobsthal–Mulatu numbers. The sequence that is the subject of this study is an extension of the Jacobsthal sequence. The remainder of this paper is organized as follows: In Section 2, we derive the Binet formula for the Jacobsthal–Mulatu numbers and explore its applications. We begin by reviewing key concepts and results related to Jacobsthal numbers, establishing initial relationships among the three sequences in the Jacobsthal family. In Section 3, we establish some identities for the Jacobsthal–Mulatu sequence. The classical identities are studied. While in Section 4 we focus on obtaining the generating functions for the Jacobsthal–Mulatu sequence. Specifically, we establish three types: the ordinary generating function, the exponential generating function, and the Poisson generating function. In Section 5, we investigate the properties and identities associated with the partial sums of the terms involving the Jacobsthal–Mulatu sequence, and moreover, the limit of some quotients is presented. Finally, in Conclusion, some notes on future research are stated.

2. Binet's Formula and Applications

In this section, we determine the Binet formula for the Jacobsthal–Mulatu numbers and present some applications. Firstly, we take up some concepts and results relating to Jacobsthal numbers and present some of the first relationships between these sequences of the Jacobsthal family.

2.1. Background and preliminary results

The Jacobsthal-type sequence is known to be associated with the characteristic equation

$$r^2 = r + 2 \tag{3}$$

whose two distinct roots are $r_1 = 2$ and $r_2 = -1$, which play a central role in deriving the properties of the sequence. For further details on Jacobsthal numbers, see [2, 3, 9, 14, 15, 16].

Lemma 2.1 ([9], Equations (2.3) and (2.4))**.** *Let $n \geq 0$. Then, the following hold:*

$$J_n = \frac{r_1^n - r_2^n}{r_1 - r_2} = \frac{2^n - (-1)^n}{3} \tag{4}$$

and

$$j_n = r_1^n + r_2^n = 2^n + (-1)^n \,. \tag{5}$$

Equations (4) and (5), respectively, are Binet's formula for the Jacobsthal and Jacobsthal–Lucas sequences. We present a similar result for the Jacobsthal–Mulatu sequence as follow:

Proposition 2.2 (Binet-like Formula)**.** *Let* $n \geq 0$*. Then, the following hold:*

$$JM_n = \frac{(1-4r_2)r_1^n + (4r_1-1)r_2^n}{r_1 - r_2} = \frac{5 \cdot 2^n + 7 \cdot (-1)^n}{3}\,. \tag{6}$$

Proof. Let r_1 and r_2 be the distinct roots of Equation (3). We need to find integers c_1 and c_2 such that

$$JM_n = c_1 r_1^n + c_2 r_2^n\,.$$

By the initial conditions,

$$\begin{cases} JM_0 = c_1 + c_2 = 4 \\ JM_1 = c_1 r_1 + c_2 r_2 = 1 \end{cases}$$

which implies that $c_1 = \frac{1-4r_2}{r_1-r_2}$ and $c_2 = \frac{4r_1-1}{r_1-r_2}$. Then,

$$JM_n = \frac{(1-4r_2)r_1^n + (4r_1-1)r_2^n}{r_1 - r_2}\,.$$

Since $r_1 = 2$ and $r_2 = -1$, the proof is concluded. □

This sequence with the general term $\frac{5 \cdot 2^n + 7 \cdot (-1)^n}{3}$ index-linked as $A344109$ in the OEIS [1] is connected to the Jacobsthal recurrence and will be referred to as Jacobsthal–Mulatu. However, we have yet to show that this sequence satisfies Equation (1). Since the Jacobsthal and Jacobsthal-Lucas sequences are not the main object of study in this work, we will list some identities involving these sequences, omitting their proofs, which will be necessary for the proof of properties of the Jacobsthal-Mulatu sequence.

Lemma 2.3 ([9], Equations (2.5), (2.6), (2.10) and (2.11))**.** *Let* $n \geq 1$*. Then, the following hold:*

$$9J_n = 2j_{n-1} + j_{n+1} \tag{7}$$

$$J_{n-1}J_{n+1} - (J_n)^2 = (-1)^n 2^{n-1} \tag{8}$$

$$j_{n-1}j_{n+1} - (j_n)^2 = 9(-1)^{n-1}2^{n-1} \tag{9}$$

and

$$j_n = 2J_{n-1} + J_{n+1}\,. \tag{10}$$

Lemma 2.4. *Let $n \geq 1$. Then, the following hold:*

$$\begin{aligned} 9J_nJ_{n+1} &= j_{2n+1} + (-1)^{n+1}2^n && (11)\\ 9(J_n)^2 &= j_{2n} + (-1)^{n+1}2^{n+1} && (12)\\ J_{n+1}\,j_n &= J_{2n+1} + (-1)^n 2^n && (13)\\ J_n\,j_{n+1} &= J_{2n+1} - (-1)^n 2^n && (14)\\ j_n\,j_{n+1} &= j_{2n+1} + (-1)^n 2^n && (15)\\ 3(J_{n+1} + J_{n-1}) &= 2j_n + 2^{n-1} && (16) \end{aligned}$$

and

$$j_{n+1} + j_{n-1} = 6J_n + 2^{n-1}. \quad (17)$$

Proof. Using Equations (4) and (5), we have

$$\begin{aligned} J_n\,j_{n+1} &= \frac{r_1^n - r_2^n}{r_1 - r_2}(r_1^{n+1} + r_2^{n+1})\\ &= \frac{1}{r_1 - r_2}\left[r_1^{2n+1} + r_2(r_1r_2)^n - (r_1r_2)^n r_1 - r_2^{2n+1}\right]\\ &= \frac{1}{r_1 - r_2}\left[r_1^{2n+1} - (r_1r_2)^n(r_1 - r_2) - r_2^{2n+1}\right]\\ &= \frac{1}{r_1 - r_2}\left[r_1^{2n+1} - r_2^{2n+1}\right] - (r_1r_2)^n = J_{2n+1} - (-1)^n 2^n, \end{aligned}$$

and then we have obtained Equation (14). Equations (11), (12), (13), (15), (17), and (16) are obtained in a similar manner. □

Afterward, we consider a generalized Jacobsthal sequence $\{GJ_n\}_{n\geq 0}$ that extends the classical Jacobsthal sequence by incorporating two arbitrary initial values GJ_0 and GJ_1. It is defined as follows:

$$GJ_n = GJ_{n-1} + 2GJ_{n-2}, \quad \text{for all } n \geq 2,$$

with the initial conditions $GJ_0 = a$ and $GJ_1 = b$, where a and b are arbitrary integers.

Proposition 2.5. *Let $n \geq 0$ and $m \geq 1$. Then, the following hold:*

$$GJ_{n+m} = 2J_{m-1} \cdot GJ_n + J_m \cdot GJ_{n+1}, \quad (18)$$

where $\{J_n\}_{n\geq 0}$ is the Jacobsthal sequence.

Proof. The proof is made using the principle of mathematical induction on m. For $m = 1$ and $m = 1$, the identity holds because $J_0 = 0$, $J_1 = 1$, and $J_2 = 1$. Suppose that Equation (18) is verified for every value less than or equal to m, for some $m \geq 2$. Then,

$$\begin{aligned} GJ_{n+m+1} &= GJ_{n+m} + 2GJ_{n+m-1} \\ &= 2J_{m-1}GJ_n + J_m GJ_{n+1} + 2 \cdot 2J_{m-2}GJ_n + 2J_{m-1}GJ_{n+1} \\ &= 2[J_{m-1} + 2J_{m-2}]GJ_n + [J_m + 2J_{m-1}]GJ_{n+1} \\ &= 2J_m GJ_n + J_{m+1}GJ_{n+1} \,. \end{aligned}$$

Therefore, the identity holds for every non-negative integer. □

A direct and immediate consequence of Proposition 2.5 is the following corollary.

Corollary 2.6. *Let $n \geq 0$ and $m \geq 1$. Then, the following hold:*

$$J_{n+m} = 2J_{m-1}J_n + J_m J_{n+1} \tag{19}$$

and

$$j_{n+m} = 2J_{m-1}j_n + J_m j_{n+1} \,. \tag{20}$$

The following auxiliary result involves two generalized Jacobsthal sequences and can be seen as a version of Equation (18) presented in [4].

Proposition 2.7. *For any generalized Jacobsthal sequences $\{GJ_n\}_{n\geq 0}$ and $\{HJ_n\}_{n\geq 0}$, that is, both satisfy* (2)*, the following identity holds:*

$$GJ_{n+m}HJ_{n+l} - GJ_n HJ_{n+m+l} = (-1)^n 2^n (GJ_m\, HJ_l - GJ_0\, HJ_{m+l}). \tag{21}$$

Proof. Define

$$I_n = GJ_{n+m}HJ_{n+l} - GJ_n HJ_{n+m+l} \,.$$

From Equation (18) , since

$$\begin{aligned} I_n &= GJ_{n+m}\, HJ_{n+l} - GJ_n\, HJ_{n+m+l} \\ &= (2J_{m-1}GJ_n + J_m GJ_{n+1})HJ_{n+l} - GJ_n(2J_{m-1}HJ_{n+l} + J_m HJ_{n+l+1}) \\ &= J_m(GJ_{n+1}\, HJ_{n+l} - GJ_n\, HJ_{n+l+1}) \end{aligned}$$

and

$$\begin{aligned} I_{n+1} &= (2J_m GJ_n + J_{m+1}GJ_{n+1})HJ_{n+l+1} - GJ_{n+1}(2J_m HJ_{n+l} + J_{m+1}HJ_{n+l+1}) \\ &= 2J_m(GJ_n\, HJ_{n+l+1} - GJ_{n+1}HJ_{n+l}), \end{aligned}$$

then

$$I_{n+1} = -2I_n \,.$$

Thus,

$$I_n = (-1)^n 2^n I_0,$$

and hence

$$GJ_{n+m}HJ_{n+l} - GJ_nHJ_{n+m+l} = (-1)^n 2^n (GJ_mHJ_l - GJ_0HJ_{m+l}) .$$

□

2.2. Connection of Jacobsthal–Mulatu numbers with Jacobsthal–type numbers

In this subsection, we explore the relationships between Jacobsthal-type sequences and the Jacobsthal–Mulatu sequence. First, it follows from Binet's formula that the following result holds:

Proposition 2.8. *Let $n \geq 1$. Then, the following hold:*

$$JM_n = J_n + 8J_{n-1} . \tag{22}$$

Proof. According to Equation (6) and the fact that $r_1r_2 = -2$, we have

$$JM_n = \frac{(1-4r_2)r_1^n - (1-4r_1)r_2^n}{r_1 - r_2} = \frac{r_1^n - r_2^n}{r_1 - r_2} + 8\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} .$$

Thus, the result follows from Equation (4) . □

In a similar way, we have the following result.

Proposition 2.9. *Let $n \geq 0$. Then, the following hold:*

$$JM_n = J_{n+1} + j_n + (-1)^n .$$

Afterward, we will show that each Jacobsthal–Mulatu number is a linear combination of Jacobsthal and Jacobsthal–Lucas numbers.

Proposition 2.10. *Let $n \geq 0$. Then, the following hold:*

$$JM_n = 2j_n - J_n . \tag{23}$$

Proof. The proof is given by mathematical induction. Note that

$$JM_0 = 2j_0 - J_0 = 2 \cdot 2 - 0 = 4,$$

and

$$JM_1 = 2j_1 - J_1 = 2 \cdot 1 - 1 = 1 .$$

Suppose that the result is valid for any positive integer less than or equal to n, for some n. Then ,

$$\begin{aligned} JM_{n+1} &= JM_n + 2JM_{n-1} \\ &= 2j_n - J_n + 2(2j_{n-1} - J_{n-1}) \\ &= 2(j_n + 2j_{n-1}) - (J_n + 2J_{n-1}) \\ &= 2j_{n+1} - J_{n+1} \,. \end{aligned}$$

Thus, the result is valid for any non-negative integer. □

Therefore, we obtain as follows that each Jacobsthal–Lucas number is the arithmetic mean Jacobsthal and Jacobsthal–Mulatu numbers:

Corollary 2.11. *Let $n \geq 0$. Then, the following hold:*

$$2j_n = J_n + JM_n \,.$$

According to Equation (1), Corollary 2.11 shows that the sequence $\{JM_n\}_{n\geq 0}$ has suffix Mulatu.

Another similar result can be obtained. We omit the proof of the following two results for brevity.

Proposition 2.12. *Let $n \geq 0$. Then, the following hold:*

$$3(JM_{n+1} + JM_{n-1}) = 36J_n - 2j_n + 5 \cdot 2^{n-1} \,.$$

Proposition 2.13. *Let $n \geq 0$. Then, the following hold:*

$$JM_{n+1} + 2JM_{n-1} = 18J_n - j_n \,. \tag{24}$$

2.3. Negative subscripts for the Jacobsthal–Mulatu sequence

In this subsection, we extend the definition of the Jacobsthal–Mulatu sequence to negative subscripts. Using standard techniques for Fibonacci-type sequences, we establish a recurrence relation for negative subscripts and derive explicit formulas consistent with the properties of the sequence.

For the Jacobsthal and Jacobsthal–Lucas numbers with negative subscripts, according to Equation (19) in [15], we have the relations

$$J_{-n} = \frac{(-1)^{n+1}}{2^n} J_n \tag{25}$$

and

$$j_{-n} = \frac{(-1)^n}{2^n} j_n \,. \tag{26}$$

To extend the Jacobsthal–Mulatu sequence to negative subscripts, we use the modified recurrence relation

$$2JM_{n-2} = JM_n - JM_{n-1}.$$

Then, we have the following pattern:

$$\begin{aligned} 2JM_{-1} &= JM_1 - JM_0 = -1(-1+1.2^2) = -1(-J_1 + J_2 \cdot 2^2)\,, \\ 2^2JM_{-2} &= 2JM_0 - 2JM_{-1} = -1 + 3 \cdot 2^2 = 1(-J_2 + J_3 \cdot 2^2)\,, \\ 2^3JM_{-3} &= 2^2JM_{-1} - 2^2JM_{-2} = 3 - 5 \cdot 2^2 = -1(-J_3 + J_4 \cdot 2^2)\,. \end{aligned}$$

Proposition 2.14. *Let $n \geq 0$ an integer. Then, the following hold:*

$$JM_{-n} = \frac{(-1)^n}{2^n}\left(2^2J_{n+1} - J_n\right)\,.$$

Proof. Let $U_{-n} = \frac{(-1)^n}{2^n}\left(2^2J_{n+1} - J_n\right)$ for all integers $n \geq 0$. We want to show that U_n verifies the recurrence relation $U_{-n} = 2U_{-(n+2)} + U_{-(n+1)}$. Indeed,

$$\begin{aligned} 2U_{-(n+2)} + U_{-(n+1)} &= 2\frac{(-1)^{n+2}}{2^{n+2}}\left(2^2J_{n+3} - J_{n+2}\right) + \frac{(-1)^{n+1}}{2^{n+1}}\left(2^2J_{n+2} - J_{n+1}\right) \\ &= \frac{(-1)^n}{2^{n+1}}2\left(2^2J_{n+1} - J_n\right) = U_{-n}\,. \end{aligned}$$

Moreover, $U_{-1} = -\frac{1}{2}(2^2J_2 - J_1)$ and $U_{-2} = \frac{1}{4}(2^2J_3 - J_2)$. So, as $\{U_n\}_{n\geq 0}$ satisfies the recurrence that defines $\{JM_n\}_{n\geq 0}$ with the same initial conditions, we conclude that $U_{-n} = JM_{-n}$. □

Another way to express $\{JM_n\}_{n\geq 0}$ is through the following result:

Proposition 2.15. *Let $n \geq 0$. Then, the following hold:*

$$JM_{-n} = \frac{(-1)^n}{2^n}\left(JM_n + 2J_n\right)\,. \tag{27}$$

Proof. Combining Equations (23), (25) and (26), the result follows. □

3. Some Properties

In this section, we establish some identities for the Jacobsthal–Mulatu sequence. The classical identities are studied, and finally, the limit of some quotients is presented.

3.1. First identities for the Jacobsthal–Mulatu sequence

We start with the result that establishes the multiplication formula for two consecutive terms of the Jacobsthal–Mulatu sequence.

Proposition 3.1. *Let $n \geq 0$. Then, the following hold:*

$$9JM_nJM_{n+1} = j_{2n+1} + 2^4j_{2n} + 2^6j_{2n-1} + 35(-1)^n2^n\,.$$

Proof. By Equation (22), we have

$$JM_nJM_{n+1} = J_nJ_{n+1} + 2^3[J_n^2 + J_{n-1}J_{n+1}] + 2^6J_{n-1}J_n\,.$$

Hence, by Equations (8), (9), (10), (11) and (12) it follows that

$$\begin{aligned}9JM_nJM_{n+1} &= j_{2n+1} - (-1)^n2^n + 2^39[(-1)^n2^{n-1} + 2(J_n)^2] + 2^6[j_{2n-1} + (-1)^n2^{n-1}]\\ &= j_{2n+1} - (-1)^n2^n + 2^3[9(-1)^n2^{n-1} + 2(j_{2n} + (-1)^{n+1}2^{n+1})] + 2^6[j_{2n-1} + (-1)^n2^{n-1}]\\ &= j_{2n+1} - (-1)^n2^n + 2^4j_{2n} + 2^6j_{2n-1} + 9(-1)^n2^{n+2} + (-1)^{n+1}2^{n+5} + (-1)^n2^{n+5}\\ &= j_{2n+1} + 2^4j_{2n} + 2^6j_{2n-1} + 35(-1)^n2^n\,,\end{aligned}$$

which proves the result. □

Proposition 3.2. *For all non-negative integer n, the sequence $\{JM_n\}_{n\geq0}$ satisfies the following identity:*

$$9JM_nJM_{n+1} = 37j_{2n+1} - 36J_{2n+1} + 35(-1)^n2^n\,. \tag{28}$$

Proof. By Equations (11), (13), (14), (15), and (23) we obtain

$$\begin{aligned}9JM_nJM_{n+1} &= 9(2j_n - J_n)(2j_{n+1} - J_{n+1})\\ &= 9[2^2j_nj_{n+1} - 2j_nJ_{n+1} - 2J_nj_{n+1} + J_{n+1}J_n]\\ &= 2^29[j_{2n+1} + (-1)^n2^n] - 2^29J_{2n+1} + [j_{2n+1} - (-1)^n2^n]\\ &= 37j_{2n+1} - 36J_{2n+1} + 35(-1)^n2^n,\end{aligned}$$

which completes the proof. □

An immediate consequence of the Proposition 3.2 is the following corollary.

Corollary 3.3. *The Jacobsthal–Mulatu sequence satisfies the identity*

$$9JM_nJM_{n+1} = 15JM_{2n+1} + 35(-1)^n2^n - 14,$$

for all non-negative integer n.

Proof. By using the Binet formula for Jacobsthal, Jacobsthal–Lucas and Jacobsthal–Mulatu sequences, we obtain

$$\begin{aligned}37j_{2n+1} - 36J_{2n+1} &= 25\cdot2^{2n+1} - 5\cdot7 - 14\\ &= 5\left(5\cdot2^{2n+1} + 7\cdot(-1)^{2n+1}\right) - 14\\ &= 15\left(\frac{5\cdot2^{2n+1} + 7\cdot(-1)^{2n+1}}{3}\right) - 14\\ &= 15\cdot JM_{2n+1} - 14.\end{aligned}$$

The result follows from Equation (28). □

We present two interesting results that exhibit combinations of certain terms of these sequences.

Proposition 3.4. *The Jacobsthal–Mulatu sequence satisfies the following identities*

$$JM_{n+m} = 2J_{m-1}JM_n + J_mJM_{n+1}\,. \tag{29}$$

for any n and m non-negative integers.

Proof. Take $GJ_i = JM_i$ in Equation (18). □

Proposition 3.5. *Let n and m be integers such that $n - m \geq 0$. Then, for the Jacobsthal–Mulatu sequence $\{JM_n\}_{n\geq 0}$, the following identity holds:*

$$2JM_{n+m} = j_mJM_n + J_m\left(18J_n - j_n\right)\,. \tag{30}$$

Proof. Changing m by $-m$ in Equation (18), and by using Equations (9) and (25), we have

$$JM_{n-m} = \frac{(-1)^m}{2^m}(J_{m+1}JM_n - J_mJM_{n+1}).$$

Then

$$\begin{aligned} JM_{n+m} + 2^m(-1)^mJM_{n-m} &= 2J_{m-1}JM_n + J_mJM_{n+1} + J_{m+1}JM_n - J_mJM_{n+1} \\ &= (2J_{m-1} + J_{m+1})JM_n \\ &= j_m \cdot JM_n \end{aligned}$$

and

$$\begin{aligned} JM_{n+m} - 2^m(-1)^mJM_{n-m} &= 2J_{m-1}JM_n + J_mJM_{n+1} - J_{m+1}JM_n + J_mJM_{n+1} \\ &= (2J_{m-1} - J_{m+1})JM_n + 2J_mJM_{n+1} \\ &= -J_mJM_n + 2J_mJM_{n+1} \\ &= J_m(2JM_{n-1} - JM_{n+1}) + 2J_mJM_{n+1} \\ &= 2J_mJM_{n-1} - J_mJM_{n+1} + 2J_mJM_{n+1} \\ &= 2J_mJM_{n-1} + J_mJM_{n+1} \\ &= J_m(2JM_{n-1} + JM_{n+1})\,. \end{aligned}$$

By summing the last two equations, we obtain

$$2JM_{n+m} = j_mJM_n + J_m(2JM_{n-1} + JM_{n+1})\,.$$

Using Equation (24), we obtain the result. □

3.2. Some classical identities

The Tagiuri-Vajda identity for the Jacobsthal-Mulatu sequence is presented as follows, which we get using the previous result.

Proposition 3.6 (Tagiuri-Vajda's identity)**.** *Let n, m and q be natural numbers. Then , the following hold:*

$$JM_{n+m} \cdot JM_{n+q} - JM_n \cdot JM_{n+m+q} = 35(-1)^{n+1}2^n J_m J_q \,. \tag{31}$$

Proof. Taking $GJ_i = H_i = JM_i$ in Equation (21) , we have

$$\begin{aligned} JM_{n+m} \cdot JM_{n+q} - JM_n \cdot JM_{n+m+q} &= (-1)^n 2^n \left[JM_m \cdot JM_q - JM_0 \cdot JM_{m+q}\right] \\ &= (-1)^n 2^n \left[JM_m \cdot JM_q - 4JM_{m+q}\right] \\ &= (-1)^n 2^n \left[JM_m \cdot JM_q - 2 \cdot 2JM_{m+q}\right] \,. \end{aligned}$$

By Equations (23) and (30), we have

$$\begin{aligned} JM_{n+m} \cdot JM_{n+q} - JM_n \cdot JM_{n+m+q} &= (-1)^n 2^n [JM_m \cdot JM_q - 2(j_q \cdot JM_m + J_q(18J_m - j_m))] \\ &= (-1)^n 2^n [JM_m \cdot JM_q - 2j_q \cdot JM_m - 36J_qJ_m + 2J_qj_m] \\ &= (-1)^n 2^n [JM_m(JM_q - 2j_q) - 2J_q(9 \cdot 2J_m - j_m)] \\ &= (-1)^n 2^n [-JM_mJ_q - 2J_q(9 \cdot 2J_m - j_m)] \\ &= (-1)^n 2^n [-J_q[JM_m + 9 \cdot 4J_m - 2j_m]] \\ &= (-1)^n 2^n [-J_q[(JM_m - 2j_m) + 9 \cdot 4J_m]] \\ &= (-1)^n 2^n [-J_q[-J_m + 9 \cdot 4J_m]] \\ &= (-1)^n 2^n [-J_mJ_q[-1 + 9 \cdot 4]] \\ &= 35(-1)^{n+1}2^n J_m J_q \,, \end{aligned}$$

which establishes the proof. □

As a direct consequence of the Tagiuri-Vajda identity, the following results are generalized to establish d'Ocagne's identity, Catalan's identity, and Cassini's identity specifically for the Jacobsthal–Mulatu sequence.

Proposition 3.7 (d'Ocagne's identity)**.** *Let h and n be non-negative integer numbers such that $h - n \geq 0$. Then , the following hold:*

$$JM_h JM_{n+1} - JM_n JM_{h+1} = 35(-1)^{n+1}2^n J_{h-n} \,.$$

Proof. Consider $m = h - n$ and $q = 1$ in Equation (31), then

$$\begin{aligned} JM_h JM_{n+1} - JM_n JM_{h+1} &= 35(-1)^{n+1}2^n J_{h-n} J_1 \\ &= 35(-1)^{n+1}2^n J_{h-n} \,. \end{aligned}$$

□

Proposition 3.8 (Catalan's identity). *Let h and n be non-negative integer numbers such that $h - n \geq 0$. Then , the following hold:*

$$JM_{n+q}JM_{n-q} - (JM_n)^2 = 35(-1)^{n+q} \cdot 2^{n-q}(J_q)^2 . \quad (32)$$

Proof. Taking $m = -q$ in Equation (31), we have that

$$JM_{n-q}JM_{n+q} - JM_nJM_n = 35(-1)^{n+1}2^n J_{-q}J_q .$$

As $J_{-q} = ((-1)^{q+1}/2^q)J_q$, we have

$$JM_{n+q}JM_{n-q} - (JM_n)^2 = 35(-1)^{n+q+2}2^{n-q}(J_q)^2 .$$

□

As a consequence of Catalan's identity, since $J_1 = 1$ and by doing $q = 1$ in Equation (32), we have the following result.

Corollary 3.9 (Cassini-Simson's identity). *For all non-negative integer n, we have*

$$(JM_n)^2 - JM_{n+1}JM_{n-1} = 35(-1)^n 2^{n-1} .$$

Other consequence is the Cassini-Simson identity for subscripts even:

Corollary 3.10. *For all non-negative integer n, we have*

$$(JM_{2n})^2 - JM_{2n+1}JM_{2n-1} = 35 \cdot 2^{2n-1} .$$

Now, we present the convolution identity for the Jacobsthal–Mulatu sequence.

Proposition 3.11 (Convolution's identity). *Let m and n be non-negative integer numbers . Then, the Jacobsthal–Mulatu sequence $\{JM_n\}_{n\geq 0}$ satisfies the following identity:*

$$JM_{m+n} = 2JM_{m-1}JM_n + JM_mJM_{n+1} - 2^3 JM_{m+n-1} .$$

Proof. By Equations (22) and (19), it follows that

$$\begin{aligned}
& 2JM_{m-1}JM_n + JM_mJM_{n+1} \\
= \;& 2(J_{m-1} + 2^3J_{m-2})(J_n + 2^3J_{n-1}) + (J_m + 2^3J_{m-1})(J_{n+1} + 2^3J_n) \\
= \;& 2J_{m-1}J_n + 2\cdot 2^3J_{m-1}J_{n-1} + 2\cdot 2^3J_{m-2}J_n + 2\cdot 2^{2\cdot 3}J_{m-2}J_{n-1} \\
& +J_mJ_{n+1} + 2^3J_mJ_n + 2^3J_{m-1}J_{n+1} + 2^{2\cdot 3}J_{m-1}J_n \\
= \;& 2J_{m-1}J_n + J_mJ_{n+1} + 2^3(2J_{m-1}J_{n-1} + J_mJ_n) \\
& +2^3(2J_{m-2}J_n + J_{m-1}J_{n+1}) + 2^{2\cdot 3}(2J_{m-2}J_{n-1} + J_{m-1}J_n) \\
= \;& J_{m+n} + 2^3J_{m+n-1} + 2^3(J_{m+n-1} + 2^3J_{m+n-2}) \\
= \;& JM_{m+n} + 2^3JM_{m+n-1}\,.
\end{aligned}$$

□

The next result follows directly from Catalan's identity.

Proposition 3.12 (Gelin-Cesaro's identity)**.** *The Jacobsthal–Mulatu sequence* $\{JM_n\}_{n\geq 0}$ *satisfies the following identity:*

$$JM_{n+2}JM_{n+1}JM_{n-1}JM_{n-2} - (JM_n)^4 = 35\cdot 2^{n-2}\left((-1)^{n+1}(JM_n)^2 - 2(35)2^{n-2}\right)\,.$$

Proof. Using Equation (32) for $q = 1$ we have

$$JM_{n+1}JM_{n-1} - (JM_n)^2 = 35(-1)^{n+1}\cdot 2^{n-1}(J_1)^2\,, \tag{33}$$

and by (32) for $q = 2$ we have

$$JM_{n+2}JM_{n-2} - (JM_n)^2 = 35(-1)^{n+2}\cdot 2^{n-2}(J_2)^2\,. \tag{34}$$

Since $J_1 = J_2 = 1$, and combining Equation s (33) and (34), we obtain

$$\begin{aligned}
& JM_{n+2}JM_{n+1}JM_{n-1}JM_{n-2} - (JM_n)^4 \\
& = \left[(JM_n)^2 + 35(-1)^{n+2}2^{n-2}(J_2)^2\right]\left[(JM_n)^2 + 35(-1)^{n+1}2^{n-1}(J_1)^2\right] - (JM_n)^4 \\
& = \left[(JM_n)^2 + 35(-1)^{n}2^{n-2}\right]\left[(JM_n)^2 - 35(-1)^{n}2^{n-1}\right] - (JM_n)^4 \\
& = (JM_n)^4 - 35(-1)^n2^{n-1}(JM_n)^2 + 35(-1)^n2^{n-2}(JM_n)^2 - (35)^2(-1)^{2n}2^{2n-3} - (JM_n)^4 \\
& = -35(-1)^n2^{n-2}(JM_n)^2 - (35)^22^{2n-3} \\
& = -35(-1)^n2^{n-2}(JM_n)^2 - 2((35)2^{n-2})^2 \\
& = 35\cdot 2^{n-2}\left((-1)^{n+1}(JM_n)^2 - 2(35)2^{n-2}\right)\,.
\end{aligned}$$

□

4. Generating Functions

In this section, we derive the generating functions for the Jacobsthal–Mulatu sequence. More specifically, we present three types of generating functions for the Jacobsthal–Mulatu sequence: the ordinary generating function, the exponential generating function, and the Poisson generating function.

It is well known that the ordinary generating function for a sequence $\{a_n\}_{n\geq 0}$, denoted as $G_{a_n}(x)$, is defined as:

$$G_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots . \tag{35}$$

According to Equations (2.1) and (2.2) in [9], the ordinary generating function of Jacobsthal and Jacobsthal–Lucas sequences has the following form

$$G_{J_n}(x) = \frac{x}{1-x-2x^2} \text{ and } G_{j_n}(x) = \frac{2-x}{1-x-2x^2} .$$

This means that we have:

$$G_{J_n}(x) = x + x^2 + 3x^3 + 5x^4 + 11x^5 + 21x^6 + 43x^7 + 85x^8 + 171x^9 + 351x^{10} + \cdots ,$$

and

$$G_{j_n}(x) = 2 + x + 5x^2 + 7x^3 + 17x^4 + 31x^5 + 65x^6 + 127x^7 + 257x^8 + 511x^9 + \cdots .$$

The following result provides the explicit form of the ordinary generating function for the Jacobsthal–Mulatu sequence.

Proposition 4.1. *The ordinary generating function for the Jacobsthal–Mulatu sequence, denoted by $G_{JM_n}(x)$, is given by*

$$G_{JM_n}(x) = \frac{4-3x}{1-x-2x^2} .$$

Proof. According to Equation (35), the ordinary generating function for the Jacobsthal–Mulatu sequence is given by

$$G_{JM_n}(x) = JM_0 + JM_1 x + JM_2 x^2 + JM_3 x^3 + \cdots + JM_n x^n + \cdots .$$

Using the relationships $xG_{JM_n}(x)$ and $2x^2 G_{JM_n}(x)$ and the fact that $JM_0 = 4$ and $JM_1 = 1$, we derive the following results:

$$G_{JM_n}(x)(1-x-2x^2) = JM_0 + [JM_1 - JM_0]x ,$$

and then

$$G_{JM_n}(x) = \frac{4-3x}{1-x-2x^2} ,$$

since $1 - x - 2x^2 \neq 0$, this completes the proof. □

The exponential generating function $E_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is represented as a power series given by:

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

In the next result, we consider the case where $a_n = JM_n$ and apply the Binet formula for the Jacobsthal–Mulatu sequence. By doing so, we derive its exponential generating function.

Proposition 4.2. *The exponential generating function for the Jacobsthal–Mulatu sequence* $\{JM_n\}_{n\geq 0}$ *is*

$$E_{JM_n}(x) = \sum_{n=0}^{\infty} JM_n \frac{x^n}{n!} = \frac{5e^{2x} + 7e^{-x}}{3}.$$

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is defined as:

$$P_{a_n}(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} e^{-x}.$$

This function encodes the sequence $\{a_n\}_{n\geq 0}$ in terms of the parameter x. A significant relationship exists between the exponential generating function $E_{a_n}(x)$ and the Poisson generating function $P_{a_n}(x)$, given by:

$$P_{a_n}(x) = e^{-x} E_{a_n}(x).$$

This relationship establishes a direct connection between the two generating functions.

As a particular case, considering the exponential generating function for the Jacobsthal–Mulatu sequence given in Proposition 4.2, we obtain the following result.

Corollary 4.3. *The Poisson generating function for the Jacobsthal–Mulatu sequence* $\{JM_n\}_{n\geq 0}$ *is:*

$$P_{JM_n}(x) = \frac{5e^{x} + 7e^{-2x}}{3}.$$

5. Sum and Ratio

In this section, firstly we explore the properties and identities related to the partial sums of terms of the Jacobsthal–Mulatu sequence. Moreover, for any positive integers n and t, the ratio between the terms a_{n+t} and a_n is given by $q_{n+t} = \frac{a_{n+t}}{a_n}$. This analysis helps to understand the growing and asymptotic pattern of the sequence.

5.1. A finite sum involving the Jacobsthal–Mulatu numbers

In this subsection, we investigate the properties and identities associated with the finite sums of the terms of the Jacobsthal–Mulatu sequence.

The sum of the first $n+1$ terms of this sequence is expressed as:

$$\sum_{k=0}^{n} JM_k = JM_0 + JM_1 + JM_2 + \cdots + JM_{n-1} + JM_n \,.$$

We begin by presenting three key results concerning the finite sums of terms of the Jacobsthal–Mulatu sequence.

First, the sum of the first $n+1$ terms.

Proposition 5.1. *For all non-negative integer n, the following hold:*

$$\sum_{k=0}^{n} JM_k = \frac{1}{2}(JM_{n+2} - 1)\,. \tag{36}$$

Proof. According to Equation (2) , we have the following equations:

$$\begin{aligned} 2JM_0 &= JM_2 - JM_1, \\ 2JM_1 &= JM_3 - JM_2, \\ &\vdots \\ 2JM_{n-1} &= JM_{n+1} - JM_n, \end{aligned}$$

and

$$2JM_n = JM_{n+2} - JM_{n+1}\,.$$

By adding both sides of these equations, we have

$$2\sum_{k=0}^{n} JM_k = JM_{n+2} - JM_1\,.$$

Since $JM_1 = 1$, we conclude the result. □

The sum involving the terms with even indexes of the Jacobsthal–Mulatu sequence can be expressed as:

Proposition 5.2. *For all non-negative integers n, the following hold:*

$$\sum_{k=0}^{\frac{n}{2}} JM_{2k} = \frac{1}{3}\left(JM_{n+2} + 3 + \frac{7}{2}n\right)\,, \quad \text{if } n \text{ is even}\,, \tag{37}$$

and

$$\sum_{k=0}^{\frac{n-1}{2}} JM_{2k} = \frac{1}{3}\left(JM_{n+1} + 3 + \frac{7}{2}(n-1)\right)\,, \quad \text{if } n \text{ is odd}\,. \tag{38}$$

Proof. If n is even, then we can write $n = 2m$ for some integer m. Then, by using the Binet formula, we have that

$$\begin{aligned} S &= JM_0 + JM_2 + \cdots + JM_{2m} \\ &= \frac{1}{3}\left[\left(5 + 5\cdot 2^2 + 5\cdot 2^4 + \cdots + 5\cdot 2^{2m}\right) + (7 + 7 + \cdots + 7)\right] \\ &= \frac{1}{9}\left[5\left(2^{2m+2} - 1\right) + 21(m+1)\right] \\ &= \frac{1}{9}\left[5\cdot 2^{2m+2} - 5 + 21 + 21m\right] \\ &= \frac{1}{9}\left[3JM_{2m+2} + 9 + 21m\right] \\ &= \frac{1}{3}\left[JM_{2m+2} + 3 + 7m\right]. \end{aligned}$$

If we consider n odd, then the result follows in a similar way. □

The sum of the terms with odd indexes of the Jacobsthal–Mulatu sequence is given as follows:

Proposition 5.3. *For all non-negative integer n, the following hold:*

$$\sum_{k=1}^{\frac{n}{2}} JM_{2k-1} = \frac{1}{6}(JM_{n+2} - 9 - 7n), \quad \textit{if } n \textit{ is even}, \tag{39}$$

and

$$\sum_{k=0}^{\frac{n-1}{2}} JM_{2k+1} = \frac{1}{6}\left(3JM_{n+2} - 2JM_{n+1} - 9 - 7(n-1)\right), \quad \textit{if } n \textit{ is odd}. \tag{40}$$

Proof. Suppose n is even. Then, from Equations (36) and (37) , it follows that

$$\begin{aligned} 6\sum_{k=1}^{\frac{n}{2}} JM_{2k-1} &= 6\sum_{k=0}^{n} JM_k - 6\sum_{k=0}^{\frac{n}{2}} JM_{2k} \\ &= 3(JM_{n+2} - 1) - 2\left(JM_{n+2} + 3 + \frac{7}{2}n\right) \\ &= 3JM_{n+2} - 3 - 2JM_{n+2} - 6 - 7n \\ &= JM_{n+2} - 9 - 7n. \end{aligned}$$

If n is odd, then by Equations (36) and (38), we have

$$\begin{aligned}
6\sum_{k=0}^{\frac{n-1}{2}} JM_{2k+1} &= 6\sum_{k=0}^{n} JM_k - 6\sum_{k=0}^{\frac{n-1}{2}} JM_{2k} \\
&= 3(JM_{n+2}-1) - 2\left(JM_{n+1} + 3 + \frac{7}{2}(n-1)\right) \\
&= 3JM_{n+2} - 3 - 2JM_{n+1} - 6 - 7n + 7 \\
&= 3JM_{n+2} - 2JM_{n+1} - 9 - 7(n-1)\,.
\end{aligned}$$

□

An immediate consequence from previous results is the result presented below. This naturally arises from the established relationships and further reinforces the conclusions derived from the Propositions 5.2 and 5.3.

Proposition 5.4. *For all non-negative integer m, we have the following identities:*

(a) $\sum_{k=0}^{m}(-1)^k JM_k = \frac{1}{6}\left(4JM_{m+1} - 3JM_{m+2} + 14(m-1) + 15\right)$*, if m is odd,*

and

(b) $\sum_{k=0}^{m}(-1)^k JM_k = \frac{1}{6}[JM_{m+2} + 2\cdot 7m + 15]$ *, if m is even.*

Proof. (a) First consider that $m = 2n+1$ is odd . Thus,

$$\begin{aligned}
\sum_{k=0}^{2n+1}(-1)^k JM_k &= JM_0 - JM_1 + JM_2 - JM_3 + \cdots + JM_{2n} - JM_{2n+1} \\
&= (JM_0 + JM_2 + \cdots + JM_{2n}) - (JM_1 + JM_3 + \cdots + JM_{2n+1}) \\
&= \sum_{k=0}^{n} JM_{2k} - \sum_{k=0}^{n} JM_{2k+1}\,.
\end{aligned}$$

According to Equations (38) and (40), it follows that:

$$\begin{aligned}
\sum_{k=0}^{m}(-1)^k JM_k &= \frac{1}{3}\left(JM_{m+1} + 3 + \frac{7}{2}(m-1)\right) - \frac{1}{6}\left(3JM_{m+2} - 2JM_{M+1} - 9 - 7(m-1)\right) \\
&= \frac{1}{3}\left(JM_{m+1} + 3 + \frac{7}{2}(m-1) - \frac{1}{2}\left(3JM_{m+2} - 2JM_{m+1} - 9 - 7(m-1)\right)\right) \\
&= \frac{1}{6}\left(2JM_{m+1} + 6 + 7(m-1) - \left(3JM_{m+2} - 2JM_{m+1} - 9 - 7(m-1)\right)\right) \\
&= \frac{1}{6}\left(4JM_{m+1} - 3JM_{m+2} + 14(m-1) + 15\right)\,.
\end{aligned}$$

(b) If m is even, consider $m = 2n$ and using Equations (37) and (39), it follows that:

$$\begin{aligned} 6\sum_{k=0}^{2n}(-1)^k JM_k &= 6(JM_0 - JM_1 + JM_2 - \ldots - JM_{2n-1} + JM_{2n}) \\ &= 6\left(\sum_{k=0}^{n} JM_{2k} - \sum_{k=1}^{n} JM_{2k-1}\right) \\ &= 2\,(JM_{2n+2} + 3 + 7n) - 1(JM_{2n+2} - 9 - 14n) \\ &= JM_{2n+2} + 28n + 15\,. \end{aligned}$$

□

5.2. Some limit identities

The quotient between two successive terms of a sequence, $\{a_n\}_{n\geq 0}$, is given by $q_n = a_{n+1}/a_n$, where q_n is the ratio of the terms a_{n+1} and a_n. For example, in the classical Jacobsthal sequence $\{J_n\}_{n\geq 0}$, $q_n = J_{n+1}/J_n$, and for sufficiently large n, q_n converges to the positive root of Equation (3), the root $r_1 = 2$.

The first result shows that the quotient q_n for the Jacobsthal–Mulatu sequence $\{JM_n\}_{n\geq 0}$ also converges to r_1 when n goes to infinity.

Proposition 5.5. *If JM_n is the n-th term of the Jacobsthal–Mulatu sequence, then*

$$\lim_{n\to\infty} \frac{JM_{n+l}}{JM_n} = (r_1)^l \tag{41}$$

and

$$\lim_{n\to\infty} \frac{JM_{-(n+l)}}{JM_{-n}} = (r_2)^l \tag{42}$$

where $r_1 = 2$ and $r_2 = -1$ are the solutions of Equation (3), and n and l are any non-negative integers.

Proof. According to Binet's formula in Equation (6), we have that

$$\frac{JM_{n+l}}{JM_n} = (r_1)^l \, \frac{(1-4r_2) - (1-4r_1)(\frac{r_2}{r_1})^{n+l}}{(1-4r_2) - (1-4r_1)(\frac{r_2}{r_1})^{n}}\,.$$

Since $|r_2/r_1| < 1$, it follows that $(r_2/r_1)^n \to 0$ when $n \to \infty$. Thus,

$$\lim_{n\to\infty} \frac{JM_{n+l}}{JM_n} = (r_1)^l \, \frac{1-4r_2}{1-4r_2} = (r_1)^l\,,$$

and hence Equation (41) follows.

Using the Equation (27), we can write

$$\frac{JM_{-(n+l)}}{JM_{-n}} = \frac{\frac{(-1)^{n+l}}{2^{n+l}}\,(JM_{n+l} + 2J_{n+l})}{\frac{(-1)^n}{2^n}\,(JM_n + 2J_n)} = \frac{(-1)^l}{2^l}\,\frac{JM_{n+l} + 2J_{n+l}}{JM_n + 2J_n}\,.$$

It follows from Binet's formula that

$$\frac{JM_{n+l}+2J_{n+l}}{JM_n+2J_n}=(r_1)^l\,\frac{(1-4r_2)-(1-4r_1)(\frac{r_2}{r_1})^{n+l}+2(1-(\frac{r_2}{r_1})^{n+l})}{(1-4r_2)-(1-4r_1)(\frac{r_2}{r_1})^{n}+2(1-(\frac{r_2}{r_1})^{n})}\,.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{JM_{n+l}+2J_{n+l}}{JM_n+2J_n}\right]&=(r_1)^l\lim_{n\to\infty}\frac{(1-4r_2)-(1-4r_1)(\frac{r_2}{r_1})^{n+l}+2(1-(\frac{r_2}{r_1})^{n+l})}{(1-4r_2)-(1-4r_1)(\frac{r_2}{r_1})^{n}+2(1-(\frac{r_2}{r_1})^{n})}\\&=(r_1)^l\,\frac{(1-4r_2)+2}{(1-4r_2)+2}=(r_1)^l=2^l\,,\end{aligned}$$

that concludes the proof since $r_1=2$ and $r_2=-1$. □

In what follows, we can immediately establish the following result using fundamental tools from the calculus of limits, along with Equations (41) and (42).

Corollary 5.6. *If JM_n is the n-th term of the Jacobsthal–Mulatu sequence, then*

$$\lim_{n\to\infty}\frac{JM_n}{JM_{n+l}}=\left(\frac{1}{r_1}\right)^l$$

and

$$\lim_{n\to\infty}\frac{JM_{-n}}{JM_{-(n+l)}}=(r_2)^l$$

where $r_1=2$ and $r_2=-1$ are solutions of Equation (3), and n and l are any non-negative integers.

6. Conclusion

The Jacobsthal sequence has some other sequences as variations that have the same recurrence, the most prominent of which is the Jacobsthal-Lucas sequence. In this article, we present the Jacobsthal-Mulatu numbers by combining the Jacobsthal recurrence and modifying the initial terms of the Jacobsthal sequence to 4 and 1. This work was motivated by the aim of establishing the existence of a Jacobsthal-type sequence that exhibits properties analogous to those of the Fibonacci-Mulatu sequence, and its connections to both the Fibonacci and Fibonacci-Lucas sequence. To our surprise, it is the numerical sequence $A344109$ in OEIS [1]. Thus, this investigation aimed to introduce the Jacobsthal-Mulatu sequence and display some connections with the Jacobsthal and Jacobsthal-Lucas sequences, investigating several of their fundamental properties. Additionally, the generating functions and Binet formulas were provided, as well as, some properties of these sequences were established. Some applications of second, third and fourth order sequences are listed in Horadam [17], in particular, the Jacobsthal sequence has application to special matrices, while third order Jacobsthal numbers and Tribonacci numbers to quaternions. In future work, we plan to explore a generalization of the Jacobsthal sequence, defined by setting the initial terms as 2^k and 1, which unifies and extends both the Jacobsthal–Lucas and Jacobsthal–Mulatu sequences, a matrix approach to this sequence, as well as a combinatorial model of the Jacobsthal-Mulatu sequence, in connection with the Jacobsthal or Jacobsthal-Lucas sequences

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