Counting the number of non-isomorphic top generalized local cohomology modules

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Abstract. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ be a proper ideal of R and M and N be two finitely generated R-modules. In this paper, we give some results in order to count the number of non-isomorphic top generalized local cohomology modules, namely $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, N)$, where $\dim_R N = n < \infty$ and $\mathrm{pd}_R M = d < \infty$. We prove that this number is equal to $2^{|\operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M,N))|}$, when $\dim_R R = d + n$ and R is Cohen-Macaulay and complete with respect to the $\mathfrak{m}$-adic topology.*

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1. Introduction

Throughout $(R, \mathfrak{m})$ denote a commutative Noetherian local ring, $\mathfrak{a}$ and $\mathfrak{b}$ are proper ideals of R and M and N are two finitely generated R-modules. Assume that $\dim_R R = r$, $\dim_R N = n < \infty$, and $\mathrm{pd}_R M = d < \infty$, where $\mathrm{pd}_R M$ denote the projective dimension of M.

Recall that for an R-module L, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime of L if $\mathfrak{p} = \operatorname{Ann}_R(L/K)$ for some submodule K of L. We denote the set of attached primes of L by $\operatorname{Att}_R(L)$. This definition agrees with the usual definition of attached primes if L has a secondary representation, see [1].

In this paper, we are interested in the structure of

$$\mathrm{H}^i_{\mathfrak{a}}(M, N) := \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N),$$

the ith generalized local cohomology module with respect to $\mathfrak{a}$ and R-modules M and N, introduced by Herzog [2]. If $M = R$, the definition of generalized local cohomol-

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ogy modules reduces to the notion introduced by Grothendieck [3] of local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(N)$.

As we will see in Proposition 2.1, if L is an R-module such that $\dim_R L = n$, $\mathfrak{a} \subseteq \mathfrak{b}$ and $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, L) \neq 0$, then there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{b}}(M, L) \to \mathrm{H}^{d+n}_{\mathfrak{a}}(M, L)$. Based on this result one can ask: when is this epimorphism an isomorphism and when is it not? In other words:

Question 1.1. Let $\mathrm{pd}_R M = d < \infty$ and $\dim_R N = n < \infty$. Is it possible to count the number of non-isomorphic top local cohomology modules $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, N)$?

The purpose of this paper is to study this question and give a number to this in a particular case. In order to do this, we define the following notation to the sets

$$\mathrm{Assh}(M, N) := \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{m}}(M, N))$$

and

$$\mathrm{Assr}(M, N) := \mathrm{Att}_R(\mathrm{H}^{r}_{\mathfrak{m}}(M, N)).$$

Then we prove, on Proposition 2.4, that the number of non-isomorphic top generalized local cohomology modules $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, N)$ is less than or equal to $2^{|\,\mathrm{Assh}(M,N)|}$. It is well-known that when $M = R$ and $(R, \mathfrak{m})$ is complete with respect to the $\mathfrak{m}$-adic topology, the equality holds (it has been proved in [4, Corollary 2.9]). When M is not necessarily equal to R, we show in Corollary 3.7 that the equality holds, when the ring R is also Cohen-Macaulay.

For conventions of notation, basic results, and terminology not given in this paper, the reader can consult [5] and [6].

2. Some Results

Proposition 2.1. *Let L be an R-module such that $\dim_R L = n$. Suppose $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, L) \neq 0$ and $\mathfrak{a} \subseteq \mathfrak{b}$. Then there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{b}}(M, L) \to \mathrm{H}^{d+n}_{\mathfrak{a}}(M, L)$.*

Proof. We may assume that $\mathfrak{a} \neq \mathfrak{b}$ and choose $x \in \mathfrak{b} \setminus \mathfrak{a}$. By [7, Lemma 3.1], there is an exact sequence

$$\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M, L) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M, L) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x, L_x),$$

where L_x is the localization of L at $\{x^i \mid i \geq 0\}$.

Note that $\dim_{R_x}(L_x) < n$ and so $\mathrm{H}^{d+n}_{\mathfrak{a}R_x}(M_x, N_x) = 0$. Thus, there is an epimorphism $\mathrm{H}^{d+n}_{\mathfrak{a}+xR}(M, N) \longrightarrow \mathrm{H}^{d+n}_{\mathfrak{a}}(M, N)$. Now the assertion follows by assuming $\mathfrak{b} = \mathfrak{a} + (x_1, \ldots, x_r)$ and applying the argument for finite steps. □

Theorem 2.2 ([8, Theorem 3.3]). $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \cong \mathrm{H}_{\mathfrak{b}}^{d+n}(M,N)$ *whenever* $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) = \mathrm{Att}_R(\mathrm{H}_{\mathfrak{b}}^{d+n}(M,N))$.

The following Proposition is analogous to Theorem 2.2 but involving the dimension of R and it is worth mentioning. The proof is similar to the Theorem, so we omit it here.

Proposition 2.3. *Assume* $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{r}(M,N)) \subseteq \mathrm{Ass}_R(N)$. *In this case, there is an isomophism* $\mathrm{H}_{\mathfrak{a}}^{r}(M,N) \cong \mathrm{H}_{\mathfrak{b}}^{r}(M,N)$, *if* $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{r}(M,N)) = \mathrm{Att}_R(\mathrm{H}_{\mathfrak{b}}^{r}(M,N))$.

Proposition 2.4. *The number of non-isomorphic top generalized local cohomology modules* $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ *is less than or equal to* $2^{|\,\mathrm{Assh}(M,N)|}$.

Proof. By Proposition 2.1, $\mathrm{Att}_R(H_{\mathfrak{a}}^{n+d}(M,N)) \subseteq \mathrm{Assh}(M,N)$. Then the result follows from the Theorem 2.2. □

Our goal is to know: when does the equality hold?

To what follows, remember: $\mathrm{cd}(\mathfrak{a}, M, N) = \sup\{i \mid \mathrm{H}_{\mathfrak{a}}^{i}(M,N) \neq 0\}$.

Corollary 2.5. *Assume* $|\,\mathrm{Assh}(M,N)| = 1$. *For any ideal* $\mathfrak{a}$ *of* R, *if* $\mathrm{cd}(\mathfrak{a}, M, N) = d+n$, *then* $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \cong \mathrm{H}_{\mathfrak{m}}^{d+n}(M,N)$. *In particular* $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ *is an Artinian* R*-module.*

Proof. Note that $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N) \neq 0$, since $\mathrm{cd}(\mathfrak{a}, M, N) = d+n$. Moreover, $\mathrm{H}_{\mathfrak{m}}^{d+n}(M,N) \neq 0$, since $|\,\mathrm{Assh}(M,N)| = 1$ (if $\mathrm{H}_{\mathfrak{m}}^{d+n}(M,N) = 0$, then $\mathrm{Assh}(M,N) = \emptyset$). As the number of non-isomorphic top generalized local cohomology modules $\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)$ is not greater than 1, we have the result.

□

The next result will be a key point of the main and last result of this section.

Proposition 2.6. *Assume that* $n \geq 1$ *and* $T \subset \mathrm{Assh}(M,N)$ *is a non-empty set. Set* $\mathrm{Assh}(M,N) \setminus T = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. *Consider the following statements:*

(i) there exists an ideal $\mathfrak{a}$ *of* R *such that* $\mathrm{Att}_R(\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) = T$;

(ii) for each $1 \leq i \leq r$, *there exists* $Q_i \in \mathrm{Supp}_R(N)$ *such that* $\dim_R(R/Q_i) = 1$ *with* $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$ *and* $\mathfrak{q}_i \subseteq Q_i$.

Then (i) *implies* (ii).

Proof. As $T = \mathrm{Att}_R((\mathrm{H}_{\mathfrak{a}}^{d+n}(M,N)) \subseteq \mathrm{Att}_R((\mathrm{H}_{\mathfrak{a}}^{n}(N))$, by [8, Proposition 2.2 (ii)], then $\mathrm{H}_{\mathfrak{a}}^{n}(R/\mathfrak{p}) \neq 0$, for all $\mathfrak{p} \in T$. By the Lichtenbaum-Hartshorne Theorem ([3, 8.2.1]), this is equivalent to say that $\mathfrak{a} + \mathfrak{p}$ is a $\mathfrak{m}$-primary ideal, for all $\mathfrak{p} \in T$.

On the other hand, for each $1 \leq i \leq r$, $\mathfrak{q}_i \notin T$, which is equivalent to say that $\mathfrak{a} + \mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal (again by the Lichtenbaum-Hartshorne Theorem, [3, 8.2.1]). Then, there exists a prime ideal $Q_i \in \mathrm{Supp}_R(N)$ such that $\dim_R(R/Q_i) = 1$ and $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$. In this case, we have $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_i$. □

Now, consider the following sentence:

(i') There exists an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq T$.

In the previous hypothesis, we can show (ii) $\Rightarrow$ (i'), where $\mathfrak{a} = \bigcap_{i=1}^{r} Q_i$, as follows.

Proof. Set $\mathfrak{a} = \bigcap_{i=1}^{r} Q_i$. For each $1 \leq i \leq r$, $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$ is equivalent to say that $\mathfrak{a} + \mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal, hence by the Lichtenbaum-Hartshorne Theorem ([3, 8.2.1]), $\mathrm{H}^n_{\mathfrak{a}}(R/Q_i) = 0$. Then $\mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(N)) \subseteq T$. Therefore, $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) \subseteq T$. □

Corollary 2.7. *If* $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \neq 0$, *then there is an ideal* $\mathfrak{b} \subseteq R$ *such that* $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{b}}(M,N)$ *and* $\dim_R R/\mathfrak{b} \leq 1$.

Proof. If $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) = \mathrm{Assh}(M,N)$, then, by Theorem 2.2, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{m}}(M,N)$.

Otherwise, $d+n \geq 1$ and $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N))$ is a proper subset of $\mathrm{Assh}(M,N)$. Set $\mathrm{Assh}(M,N) \setminus \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) := \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. By the proof of Proposition 2.6, for each $1 \leq i \leq r$, there exists $Q_i \in \mathrm{Supp}_R(M)$ with $\dim_R(R/Q_i) = 1$ such that $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) = \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{b}}(M,N))$, where $\mathfrak{b} = \bigcap_{i=1}^{r} Q_i$. Again, by Theorem 2.2, $\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N) \cong \mathrm{H}^{d+n}_{\mathfrak{b}}(M,N)$. Since $\dim_R R/\mathfrak{m} = 0$ and $\dim_R R/\mathfrak{b} = 1$, we have the result. □

3. The Cohen-Macaulay case

Recall that a finitely generated R-module M is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim_R M$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring*. A *maximal Cohen-Macaulay module* is a Cohen-Macaulay module M such that $\dim_R M = \dim_R R$. For more information about Cohen-Macaulay rings and modules the reader can see the book of Bruns-Herzog [5].

From now on, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian Cohen-Macaulay local ring, complete with respect to the $\mathfrak{m}$-adic topology, such that $\dim_R R = r$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. We also assume that M and N are two finitely generated R-modules such that $\mathrm{pd}_R M = d < \infty$ and $\dim_R N = n < \infty$.

Note that $\mathrm{H}^r_{\mathfrak{a}}(M,N)$ is an Artinian R-module for all R-ideal $\mathfrak{a}$ and

$$\mathrm{Att}_R(\mathrm{H}^r_{\mathfrak{a}}(M,N)) = \{\mathfrak{p} \in \mathrm{Supp}_R(N) \cap \mathrm{Ass}_R(M) \mid \dim_R(R/\mathfrak{a}+\mathfrak{p}) = 0\},$$

by [7, Theorems 3.8 and 3.9], since R is complete with respect to the $\mathfrak{m}$-adic topology.

Proposition 3.1. *Assume $r \geq 1$ and T is a proper non-empty subset of* $\operatorname{Assr}(M, N)$. *Set* $\operatorname{Assr}(M, N) \setminus T = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$. *The following statements are equivalent:*

(i) There exists an ideal $\mathfrak{a}$ of R such that $\operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}}(M, N)) = T$;

(ii) For each $1 \leq i \leq s$, there exists $Q_i \in \operatorname{Supp}_R(N)$ such that $\dim_R(R/Q_i) = 1$ with $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \nsubseteq Q_i$ and $\mathfrak{q}_i \subseteq Q_i$.

Where $\mathfrak{a} = \bigcap_{i=1}^{s} Q_i$.

Proof. (i) $\Rightarrow$ (ii) Since

$$T = \{\mathfrak{p} \in \operatorname{Supp}_R(N) \cap \operatorname{Ass}_R(M) \mid \dim_R(R/\mathfrak{a} + \mathfrak{p}) = 0\} \neq \emptyset,$$

$\mathfrak{a}+\mathfrak{p}$ is a $\mathfrak{m}$-primary ideal, for all $\mathfrak{p} \in T$. On the other hand, by the same argument, $\mathfrak{a}+\mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal, since $\mathfrak{q}_i \notin T$, for all $1 \leq i \leq s$. Then there exists $Q_i \in \operatorname{Supp}_R(N)$ such that $\dim_R(R/Q_i) = 1$ and $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$. Therefore, $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \nsubseteq Q_i$.

(ii) $\Rightarrow$ (i) Set $\mathfrak{a} = \bigcap_{i=1}^{s} Q_i$. For all $1 \leq i \leq s$, $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$, then $\mathfrak{a} + \mathfrak{q}_i$ is not a $\mathfrak{m}$-primary ideal. Thus, $\operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}}(M, N)) \subseteq T$.

On the other hand, take $\mathfrak{p} \in T \subseteq \operatorname{Assr}(M, N) = \operatorname{Supp}_R(N) \cap \operatorname{Ass}_R(M)$ and let $Q \in \operatorname{Supp}_R(N)$ such that $\mathfrak{a} + \mathfrak{p} \subseteq Q$. Then there exists $i \in \{1, \ldots, s\}$ such that $Q_i \subseteq Q$. Notice that $\mathfrak{p} \nsubseteq Q_i$, since $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \nsubseteq Q_i$, then $Q_i \neq Q$. Thus, $Q = \mathfrak{m}$. So $\mathfrak{a} + \mathfrak{p}$ is a $\mathfrak{m}$-primary ideal then $\dim_R(R/\mathfrak{a} + \mathfrak{p}) = 0$. Therefore, $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}}(M, N))$. □

Corollary 3.2. *If $r = 1$, then any subset of $\operatorname{Assr}(M, N)$ is equal to $\operatorname{Att}_R(\mathrm{H}^1_{\mathfrak{a}}(M, N))$, for some ideal $\mathfrak{a}$ of R.*

Proof. In the notation of the Proposition 3.1, it is enough to take $Q_i = \mathfrak{q}_i$, for $i = 1, \ldots, s$. □

Lemma 3.3. *Assume $r \geq 2$. Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be ideals of R. Then there exists an ideal $\mathfrak{b}$ of R such that $\operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{b}}(M, N)) = \operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}_1}(M, N)) \cap \operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}_2}(M, N))$.*

Proof. Consider $T_1 = \operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}_1}(M, N))$ and $T_2 = \operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{a}_2}(M, N))$. Suppose $T_1 \cap T_2$ is a non-empty proper subset of $\operatorname{Assr}(M, N)$. Let $\mathfrak{q} \in \operatorname{Assr}(M, N) \setminus (T_1 \cap T_2) = (\operatorname{Assr}(M, N) \setminus T_1) \cup (\operatorname{Assr}(M, N) \setminus T_2)$. By Proposition 3.1, there exists $Q \in \operatorname{Supp}_R(N)$ such that $\dim_R(R/Q) = 1$ with $\mathfrak{q} \subseteq Q$ and $\bigcap_{\mathfrak{p} \in T_1} \mathfrak{p} \nsubseteq Q$ or $\bigcap_{\mathfrak{p} \in T_2} \mathfrak{p} \nsubseteq Q$. Thus, there exists Q satisfying the above conditions such that $\bigcap_{\mathfrak{p} \in T_1 \cap T_2} \mathfrak{p} \nsubseteq Q$. Therefore, again by Proposition 3.1, there exists an ideal $\mathfrak{b}$ of R such that $\operatorname{Att}_R(\mathrm{H}^r_{\mathfrak{b}}(M, N)) = T_1 \cap T_2$. □

Lemma 3.4. *Assume $r = d + n \geq 2$. Let T be a non-empty subset of $\operatorname{Assr}(M, N) = \operatorname{Assh}(M, N)$ and $T' = \operatorname{Assh}(M, N) \setminus T = \{\mathfrak{q}\}$. If $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \nsubseteq \bigcap_{\mathfrak{b} \in \operatorname{Assh}(M, R/\mathfrak{p})} \mathfrak{b}$,* *then there exists a prime ideal* $Q \in \mathrm{Supp}_R(N)$ *such that* $\dim_R(R/Q) = 1$ *and* $\mathrm{Att}_R(\mathrm{H}^{d+n}_Q(M,N)) = T$.

Proof. First note that $\dim_R R/\mathfrak{q} = d+n = r$, since $\mathfrak{q} \in T' \subset \mathrm{Assh}(M,N)$. We prove by induction in j $(0 \leq j \leq r-1)$ that there exists a chain of prime ideals $Q_0 \subset Q_1 \subset \cdots \subset Q_j \subset \mathfrak{m}$ such that $Q_0 \in \mathrm{Assh}(M, R/\mathfrak{q})$, $\dim_R(R/Q_j) = r-j$ and $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_j$.

Now, note that there exists $Q_0 \in \mathrm{Assh}(M, R/\mathfrak{q})$ such that $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_0$, since $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq \bigcap_{\mathfrak{b}\in \mathrm{Assh}(M,R/\mathfrak{q})} \mathfrak{b}$, and $\dim_R(R/Q_0) = \dim_R R/\mathfrak{q} = r$.

Let $0 < j \leq r-1$ and suppose we already prove the existence of the chain of prime ideals $Q_0 \subset Q_1 \subset \cdots \subset Q_{j-1}$ such that $Q_0 \in \mathrm{Assh}(M, R/\mathfrak{q})$, $\dim_R(R/Q_j) = r-(j-1)$ and $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_{j-1}$. Note that $r-j+1 \geq 2$, since $r-j-1 \geq 0$. Thus, the set

$$V = \{\mathfrak{q} \in \mathrm{Supp}_R(N) \mid Q_{j-1} \subset \mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{m},\ \dim_R R/\mathfrak{q} = r-j,$$

$$\mathfrak{q}' \in \mathrm{Spec}(R),\ \dim_R R/\mathfrak{q}' = r-j-1\}$$

is non-empty and therefore, by Ratliff's weak existence Theorem ([6, Theorem 31.2]), is infinite.

Since $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_{j-1}$, we have that $Q_{j-1} \subset Q_{j-1} + \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. If $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \subseteq \mathfrak{q}$ for $\mathfrak{q} \in V$, then $\mathfrak{q}$ is a minimal prime ideal of $Q_{j-1} + \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. Thus, there is $Q_j \in V$ such that $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq Q_j$, since V is an infinite set and the amount of minimal primes of $Q_{j-1} + \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$ is finite. Therefore, taking $Q := Q_{r-1}$, by Proposition 3.1, the result is as follows. □

Corollary 3.5. *Assume* $r = d+n \geq 2$. *If* $T \subseteq \mathrm{Assh}(M,N)$ *is non-empty and* $T' = \mathrm{Assh}(M,N) \setminus T = \{\mathfrak{q}\}$, *then there is an ideal* $\mathfrak{a}$ *of* R *such that* $\mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N)) = T$.

Proof. Define, for any prime ideal $\mathfrak{b} \in \mathrm{Spec}(R)$,

$$\mathrm{ht}_N(\mathfrak{b}) = \inf\{s \mid P_0 \subseteq P_1 \subseteq \cdots \subseteq P_s \subseteq \mathfrak{b},\ \text{with } P_i \in \mathrm{Supp}_R(N)\}.$$

Note that $\mathrm{ht}_N(\mathfrak{q}) = 0$, since $\mathfrak{q} \in \mathrm{Ass}_R(N) \subseteq \mathrm{Supp}_R(N)$. Thus $\bigcap_{\mathfrak{p}\in T} \mathfrak{p} \nsubseteq \mathfrak{q} \subseteq \bigcap_{\mathfrak{b}\in \mathrm{Assh}(M,R/\mathfrak{p})} \mathfrak{b}$. Therefore, by Lemma 3.4, the result follows. □

Now we are able to show the main result of this section.

Theorem 3.6. *Assume* $r = d+n$. *If* $T \subseteq \mathrm{Assh}(M,N)$, *then there exists an ideal* $\mathfrak{a}$ *of* R *such that* $T = \mathrm{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M,N))$.

Proof. By Corollary 3.2, we may assume $r \geq 2$ and T is a non-empty proper subset of $\operatorname{Assh}(M, N)$. Let $T = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and $\operatorname{Assh}(M, N) \setminus T = \{\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_{t+s}\}$. We use induction over s. For $s = 1$, the result holds by Corollary 3.5.

Suppose that $s > 1$ and the result holds for the case $s - 1$. Consider $T_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{p}_{t+1}\}$ and $T_2 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{p}_{t+2}\}$. By induction, there exists ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of R such that $T_1 = \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}_1}(M, N))$ and $T_2 = \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}_2}(M, N))$. Therefore, by Lemma 3.3, there exists an ideal $\mathfrak{a}$ of R such that $T = T_1 \cap T_2 = \operatorname{Att}_R(\mathrm{H}^{d+n}_{\mathfrak{a}}(M, N))$. □

As an immediate consequence of Theorem 3.6, we have our main result, which counts the number of non-isomorphic top generalized local cohomology modules.

Corollary 3.7. *The number of non-isomorphic top local cohomology modules* $\mathrm{H}^{d+n}_{\mathfrak{a}}(M, N)$ *is equal to* $2^{|\operatorname{Assh}(M,N)|}$*, when* $r = d + n$*.*

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5. References

References

[1] Macdonald IG. Secondary representation of modules over a commutative ring. In: Symposia Mathematica. vol. 11; 1973. p. 23-43.

[2] Herzog J. Komplexe, Auflösungen und Dualität in der lokalen Algebra; 1974.

[3] Brodmann MP, Sharp RY. Local cohomology: an algebraic introduction with geometric applications. vol. 136. Cambridge university press; 2012.

[4] Dibaei MT, Jafari R. Top local cohomology modules with specified attached primes. In: Algebra Colloquium. vol. 15. World Scientific; 2008. p. 341-4.

[5] Bruns W, Herzog HJ. Cohen-macaulay rings. 39. Cambridge university press; 1998.

[6] Matsumura H. Commutative ring theory. vol. 8. Cambridge university press; 1989.

[7] Divaani-Aazar K, Hajikarimi A. Generalized local cohomology modules and homological Gorenstein dimensions. Communications in Algebra. 2011;39(6):2051-67.

[8] Gu Y, Chu L. Attached primes of the top generalized local cohomology modules. Bulletin of the Australian Mathematical Society. 2009;79(1):59-67.