A Universal Algebraic Survey of $\mathcal{C}^{\infty}-$Rings

Jean Cerqueira Berni[1], **Hugo Luiz Mariano**[2]

[1]Department of Mathematics – Institute of Mathematics and Statistics
University of São Paulo
Rua do Matão, 1010, São Paulo - SP – Brazil

[2]Department of Mathematics – Institute of Mathematics and Statistics
University of São Paulo
Rua do Matão, 1010, São Paulo - SP – Brazil

jeancb@ime.usp.br,hugomar@ime.usp.br

Abstract. *In this paper we present some basic results of the Universal Algebra of $\mathcal{C}^{\infty}-$rings which were nowhere to be found in the current literature. The outstanding book of I. Moerdijk and G. Reyes, [1], presents the basic (and advanced) facts about $\mathcal{C}^{\infty}-$rings, however such a presentation has no universal algebraic "flavour". We have been inspired to describe $\mathcal{C}^{\infty}-$rings through this viewpoint by D. Joyce in [2]. Our main goal here is to provide a comprehensive material with detailed proofs of many known "taken for granted" results and constructions used in the literature about $\mathcal{C}^{\infty}-$rings and their applications - such proofs either could not be found or were merely sketched. We present, in detail, the main constructions one can perform within this category, such as limits, products, homomorphic images, quotients, directed colimits, free objects and others, providing a "propaedeutic exposition" for the reader's benefit.*
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Resumo. *Neste artigo apresentamos alguns resultados básicos da Álgebra Universal dos anéis $\mathcal{C}^{\infty}$ que não aparecem explicitamente em nenhuma parte da literatura especializada. O excelente livro de I. Moerdijk e G. Reyes, [1], apresenta fatos básicos (e avançados) sobre anéis $\mathcal{C}^{\infty}$, no entanto sem um "sabor" algébrico-universal. Nossa motivação para descrever os anéis $\mathcal{C}^{\infty}$ sob este ponto de vista foi o trabalho de D. Joyce em [2]. Nosso principal objetivo aqui é fornecer um material abrangente com demonstrações detalhadas de diversos resultados sobre anéis $\mathcal{C}^{\infty}$ tidos como garantidos na literatura sobre esses anéis*

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e aplicações — provas que não estão sequer esboçadas. Apresentamos, detalhadamente, as principais construções que se pode fazer dentro dessa categoria, tais como limites, produtos, imagens homomorfas, quocientes, limites dirigidos, objetos livres e outros, fornecendo uma "exposição propedêutica" em benefício do leitor.

Palavras-chave – *Anéis $\mathcal{C}^\infty$, construções algébricas, álgebra universal.*

Introduction

As observed by E. Dubuc in [3], a $\mathcal{C}^\infty-$ring is a model of the algebraic theory which has as $n-$ary operations all the smooth functions from $\mathbb{R}^n$ into $\mathbb{R}$, and whose axioms are all the equations that hold between these functions. Since every polynomial is smooth, all $\mathcal{C}^\infty-$rings are, in particular, $\mathbb{R}-$algebras. This point of view allows us to regard this theory as an extension of the concept of $\mathbb{R}-$algebra.

The theory of $\mathcal{C}^\infty-$rings has been originally studied in view of its applications to Singularity Theory and in order to construct topos-models for Synthetic Differential Geometry (the Dubuc Topos, for instance. See [4]), which "grew out of ideas of Lawvere in the 1960s" (cf. [2]). Recently, however, this theory has been explored by some eminent mathematicians like David I. Spivak and Dominic Joyce in order to extend Jacob Lurie's program of Derived Algebraic Geometry to Derived Differential Geometry (cf. [2] and [5]).

Just as any Lawvere theory, $\mathcal{C}^\infty-$rings can be interpreted within any topos. In this specific case, a $\mathcal{C}^\infty-$ring in a topos $\mathcal{E}$ is a finite product preserving functor from the category whose objects are the Cartesian products of $\mathbb{R}$ and whose morphisms are the smooth functions between them, into $\mathcal{E}$ (see, for example, [6]). In this work, however, we focus on set-theoretic $\mathcal{C}^\infty-$rings (i.e., $\mathcal{C}^\infty-$rings in the topos of sets, **Set**).

Some categorial and logical aspects of $\mathcal{C}^\infty-$rings were given in [7] and an order-theoretic analysis (detailing some aspects in [6]) is provided in [8]. Here we present and analyze a $\mathcal{C}^\infty-$ring as a universal algebra whose functional symbols are the symbols for all smooth functions from cartesian powers of $\mathbb{R}$ to $\mathbb{R}$. Such an approach emphasizes the power of a $\mathcal{C}^\infty-$ring in interpreting a broader language than the algebraic one, which is expressed in terms of the $\mathbb{R}-$algebra structure. It also has the advantage of giving us explicitly many constructions, such as products, coproducts, directed colimits, among others, as well as simpler proofs of the main results, which can be found in [9].

We make a detailed exposition of the description of free $\mathcal{C}^\infty-$rings in terms of a colimit, and we use it to account the often used description of an arbitrary $\mathcal{C}^\infty-$rings in terms of generators and relations.

Our idea of describing $\mathcal{C}^{\infty}-$rings from a universal-algebraic point of view was mainly inpired by the clear and elegant presentation made by Dominic Joyce in [2] - which we found very enlightening.

Overview of the paper: We begin by presenting the equational theory of $\mathcal{C}^{\infty}-$rings in terms of a first order language with a denumerable set of variables. We define the class of $\mathcal{C}^{\infty}-$structures and the (equationally defined) subclass of $\mathcal{C}^{\infty}-$rings.

In the **Section 2** we present a detailed description of the main constructions involving $\mathcal{C}^{\infty}-$rings: $\mathcal{C}^{\infty}-$subrings (**Definition 2.1**), intersections (**Proposition 2.2**), the $\mathcal{C}^{\infty}-$subring generated by a set (**Definition 2.3**), the directed union of $\mathcal{C}^{\infty}-$rings (**Proposition 2.4**), products (**Definition 2.5**), $\mathcal{C}^{\infty}-$congruences (**Definition 2.7**) and quotients (**Definition 2.10**), homomorphic images (**Proposition 2.16**), directed colimits (**Theorem 2.19**) and small projective limits (**Theorem 2.20**). We also present the "Fundamental Theorem of $\mathcal{C}^{\infty}-$Homomorphism" (**Theorem 2.15**) and we present a result which states that the category of $\mathcal{C}^{\infty}-$rings is a reflective subcategory of the category of all $\mathcal{C}^{\infty}-$structures (**Theorem 2.23**).

We dedicate **Section 3** to describe the free $\mathcal{C}^{\infty}-$rings, first with a finite set of generators and then with an arbitrary set of generators. We use this construction in order to describe an adjunction between the category of all $\mathcal{C}^{\infty}-$rings and $\mathcal{C}^{\infty}-$homomorphisms, $\mathcal{C}^{\infty}$**Ring** and the category of sets, **Set** (**Proposition 3.4**).

In **Section 4** we describe other constructions. We present a result which states that the ring-theoretic ideals of any finitely generated $\mathcal{C}^{\infty}-$ring classify their congruences (**Proposition 4.6**), and we extend this result by presenting a proof for the general case (**Proposition 4.15**).

We give a result which states that any $\mathcal{C}^{\infty}-$ring can be expressed as a directed colimit of finitely generated $\mathcal{C}^{\infty}-$rings (**Theorem 3.5**) and in **Subsection 4.2** we present an explicit description for the $\mathcal{C}^{\infty}-$coproduct of $\mathcal{C}^{\infty}-$rings (**Definition 4.19**). We end up this work presenting an ubiquitous construction in Algebra in the **Subsection 4.3**, namely the $\mathcal{C}^{\infty}-$ring of $\mathcal{C}^{\infty}-$polynomials, constructed in terms of the $\mathcal{C}^{\infty}-$product. Such a construction play an important role in [10] and [11].

1. Preliminaries: The equational theory of C^{∞}-rings

The theory of $\mathcal{C}^{\infty}-$rings can be described within a first order language $\mathcal{L}$ with a denumerable set of variables ($\mathrm{Var}\,(\mathcal{L}) = \{x_1, x_2, \cdots, x_n, \cdots\}$) whose nonlogical symbols are the symbols of $\mathcal{C}^{\infty}-$functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, with $m, n \in \mathbb{N}$, *i.e.*, the non-logical symbols consist only of function symbols, described as follows:

For each $n \in \mathbb{N}$, the $n-$ary **function symbols** of the set $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$, *i.e.*, $\mathcal{F}_{(n)} =$

$\{f^{(n)} | f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\}$. Thus, the set of function symbols of our language is given by:

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{(n)} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n)$$

Note that our set of constants is $\mathbb{R}$, since it can be identified with the set of all $0-$ary function symbols, *i.e.*, **Const**$(\mathcal{L}) = \mathcal{F}_{(0)} = \mathcal{C}^\infty(\mathbb{R}^0) \cong \mathcal{C}^\infty(\{*\}) \cong \mathbb{R}$.

The terms of this language, $\mathbb{T}$, are defined, in the usual way, as the smallest set which comprises the individual variables, constant symbols and $n-$ary function symbols followed by n terms ($n \in \mathbb{N}$).

Since $\mathcal{L}$ contains no relational symbols, the set of the **atomic formulas**, **AF**, is given simply by the equality between terms, that is

$$\mathbf{AF} = \{t_1 = t_2 | t_1, t_2 \in \mathbb{T}\}$$

Finally, the **well formed formulas**, **WFF** are constructed as one usually does in any first order theory.

Definition 1.1. A $\mathcal{C}^\infty-$**structure** on a set A is a pair $\mathfrak{A} = (A, \Phi)$, where:

$$\begin{array}{rccc} \Phi : & \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n \in \mathbb{N}} \mathrm{Func}\,(A^n; A) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \Phi(f) := (f^A : A^n \to A) \end{array},$$

that is, Φ interprets the **symbols**[1] of all smooth real functions of n variables as $n-$ary function symbols on A.

Definition 1.2. Let (A, Φ) and (B, Ψ) be two $\mathcal{C}^\infty-$structures. A function $\varphi : A \to B$ is called a **morphism of** $\mathcal{C}^\infty-$**structures** (or, simply, a $\mathcal{C}^\infty-$morphism) if for any $n \in \mathbb{N}$ and any $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ the following diagram commutes:

$$\begin{tikzcd} A^n \arrow[r, "\varphi^{(n)}"] \arrow[d, "\Phi(f)"'] & B^n \arrow[d, "\Psi(f)"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

i.e., $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$.

Theorem 1.3. *Let* $(A, \Phi), (B, \Psi), (C, \Omega)$ *be any* $\mathcal{C}^\infty-$*structures, and let* $\varphi : (A, \Phi) \to (B, \Psi)$ *and* $\psi : (B, \Psi) \to (C, \Omega)$ *be two morphisms of* $\mathcal{C}^\infty-$*structures. We have:*

(1) $\mathrm{id}_A : (A, \Phi) \to (A, \Phi)$ *is a morphism of* $\mathcal{C}^\infty-$*structures;*

(2) $\psi \circ \varphi : (A, \Phi) \to (C, \Omega)$ *is a morphism of* $\mathcal{C}^\infty-$*structures.*

[1] here considered simply as syntactic symbols rather than functions.

Proof. See **Theorem 1**, p. 5 of [9]. □

Theorem 1.4. *Let $(A,\Phi),(B,\Psi),(C,\Omega),(D,\Gamma)$ be any $\mathcal{C}^\infty-$structures, and let $\varphi : (A,\Phi) \to (B,\Psi)$, $\psi : (B,\Psi) \to (C,\Omega)$ and $\nu : (C,\Omega) \to (D,\Gamma)$ be morphisms of $\mathcal{C}^\infty-$structures. We have the following equations between pairs of morphisms of $\mathcal{C}^\infty-$structures:*

$$\nu \circ (\psi \circ \varphi) = (\nu \circ \psi) \circ \varphi;$$

$$\varphi \circ \mathrm{id}_A = \mathrm{id}_B \circ \varphi.$$

Proof. See **Theorem 2**, p. 6 of [9]. □

Definition 1.5. We are going to denote by $\mathcal{C}^\infty$**Str** the category whose objects are the $\mathcal{C}^\infty-$structures and whose morphisms are the morphisms of $\mathcal{C}^\infty-$structures.

As a full subcategory of $\mathcal{C}^\infty$**Str** we have the category of $\mathcal{C}^\infty-$rings. We call a $\mathcal{C}^\infty-$struture $\mathfrak{A} = (A,\Phi)$ a $\mathcal{C}^\infty-$**ring** if it preserves projections and all equations between smooth functions. More precisely:

Definition 1.6. Let $\mathfrak{A} = (A,\Phi)$ be a $\mathcal{C}^\infty-$structure. We say that $\mathfrak{A}$ (or, when there is no danger of confusion, A) is a $\mathcal{C}^\infty-$**ring** if the following is true:

- Given any $n,k \in \mathbb{N}$ and any projection $p_k : \mathbb{R}^n \to \mathbb{R}$, we have:

$$\mathfrak{A} \models (\forall x_1)\cdots(\forall x_n)(p_k(x_1,\cdots,x_n) = x_k)$$

- For every $f, g_1, \cdots g_n \in \mathcal{C}^\infty(\mathbb{R}^m,\mathbb{R})$ with $m,n \in \mathbb{N}$, and every $h \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ such that $f = h \circ (g_1,\cdots,g_n)$, one has:

$$\mathfrak{A} \models (\forall x_1)\cdots(\forall x_m)(f(x_1,\cdots,x_m) = h(g(x_1,\cdots,x_m),\cdots,g_n(x_1,\cdots,x_m)))$$

Definition 1.7. Let (A,Φ) and (B,Ψ) be two $\mathcal{C}^\infty-$rings. A function $\varphi : A \to B$ is called a **morphism of** $\mathcal{C}^\infty-$**rings** or $\mathcal{C}^\infty$**-homomorphism** if for any $n \in \mathbb{N}$ and any $f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}$ the following diagram commutes:

$$\begin{tikzcd} A^n \arrow[r, "\varphi^{(n)}"] \arrow[d, "\Phi(f)"'] & B^n \arrow[d, "\Psi(f)"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

i.e., $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$.

One possible set of axioms for the theory of the $\mathcal{C}^\infty$-rings can be given by the following two sets of equations:

(E1) For each $n \in \mathbb{N}$ and for every $k \leq n$, denoting the k-th projection by $p_k : \mathbb{R}^n \to \mathbb{R}$, the equations:

$$\mathbf{Eq}^{n,k}_{(1)} = \{(\forall x_1) \cdots (\forall x_n)(p_k(x_1, \cdots, x_n) = x_k)\}$$

(E2) for every $k, n \in \mathbb{N}$ and for every $(n+2)-$tuple of function symbols, $(f, g_1, \cdots, g_n, h)$ such that $f \in \mathcal{F}_{(n)}$, $g_1, \cdots, g_n, h \in \mathcal{F}_{(k)}$ and $h = f \circ (g_1, \cdots, g_n)$, the equations:

$$\mathbf{Eq}^{n,k}_{(2)} = \{(\forall x_1) \cdots (\forall x_k)(h(x_1, \cdots, x_k) = f(g_1(x_1, \cdots, x_k), \cdots, g_n(x_1, \cdots, x_k)))\}$$

As we are going to see later on, the category of $\mathcal{C}^\infty-$rings and its morphisms has many constructions, such as arbitrary products, coproducts, directed colimits, quotients and many others. It also “extends” the category of commutative unital rings, **CRing**, in the following sense:

Remark 1.8. Since the sum $+ : \mathbb{R}^2 \to \mathbb{R}$, the opposite, $- : \mathbb{R} \to \mathbb{R}$, $\cdot : \mathbb{R}^2 \to \mathbb{R}$ and the constant functions $0 : \mathbb{R} \to \mathbb{R}$ and $1 : \mathbb{R} \to \mathbb{R}$ are particular cases of $\mathcal{C}^\infty-$functions, any $\mathcal{C}^\infty-$ring (A, Φ) may be regarded as a commutative unital ring $(A, \Phi(+), \Phi(\cdot), \Phi(-), \Phi(0), \Phi(1))$, where:

$$\begin{array}{rccc} \Phi(+): & A \times A & \to & A \\ & (a_1, a_2) & \mapsto & \Phi(+)(a_1, a_2) = a_1 + a_2 \end{array}$$

$$\begin{array}{rccc} \Phi(-): & A & \to & A \\ & a & \mapsto & \Phi(-)(a) = -a \end{array}$$

$$\begin{array}{rccc} \Phi(0): & A^0 & \to & A \\ & * & \mapsto & \Phi(0) \end{array}$$

$$\begin{array}{rccc} \Phi(1): & A^0 & \to & A \\ & * & \mapsto & \Phi(1) \end{array}$$

where $A^0 = \{*\}$, and:

$$\begin{array}{rccc} \Phi(\cdot): & A \times A & \to & A \\ & (a_1, a_2) & \mapsto & \Phi(\cdot)(a_1, a_2) = a_1 \cdot a_2 \end{array}$$

Thus, we have a forgetful functor:

$$\begin{array}{rccc} \widetilde{U}: & \mathcal{C}^\infty\mathbf{Ring} & \to & \mathbf{CRing} \\ & (A\,\Phi) & \mapsto & (A,\Phi(+),\Phi(\cdot),\Phi(-),\Phi(0),\Phi(1)) \\ & \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi\upharpoonright} \\ & (B,\Psi) & & (B,\Psi(+),\Psi(\cdot),\Psi(-),\Psi(0),\Psi(1)) \end{array}$$

$$\widetilde{U}:\mathcal{C}^\infty\mathbf{Ring}\to\mathbf{CRing}$$

Analogously, we can define a forgetful functor from the category of $\mathcal{C}^\infty-$rings and $\mathcal{C}^\infty-$homomorphisms into the category of commutative $\mathbb{R}-$algebras with unity,

$$\hat{U}:\mathcal{C}^\infty\mathbf{Ring}\to\mathbb{R}-\mathbf{Alg}$$

These functors are analyzed with detail in [10].

2. The main constructions in the category of $\mathcal{C}^\infty-$rings

Since the theory of $\mathcal{C}^\infty-$rings is equational, the class $\mathcal{C}^\infty\mathbf{Ring}$ is closed in $\mathcal{C}^\infty\mathbf{Str}$ under many algebraic constructions, such as substructures, products, quotients, directed colimits and others. In this section we give explicit descriptions for some of these constructions.

2.1. $\mathcal{C}^\infty-$Subrings

We begin defining what we mean by a $\mathcal{C}^\infty-$subring.

Definition 2.1. Let (A,Φ) be a $\mathcal{C}^\infty-$ring and let $B\subseteq A$. Under these circumstances, we say that (B,Φ') is a $\mathcal{C}^\infty-$subring of (A,Φ) if, and only if, for any $n\in\mathbb{N}$, $f\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ and any $(b_1,\cdots,b_n)\in B^n$ we have:

$$\Phi(f)(b_1,\cdots,b_n)\in B$$

That is to say that B is closed under any $\mathcal{C}^\infty-$function n-ary symbol. Note that the $\mathcal{C}^\infty-$structure of B is virtually the same as the $\mathcal{C}^\infty-$structure of (A,Φ), since they interpret every smooth function in the same way. However Φ' has a different codomain, as:

$$\begin{array}{rccc} \Phi': & \bigcup_{n\in\mathbb{N}}\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}}\mathrm{Func}\,(B^n,B) \\ & (\mathbb{R}^n\xrightarrow{f}\mathbb{R}) & \mapsto & (\Phi(f)\upharpoonright_{B^n}: B^n\to B) \end{array}$$

We observe that Φ' is the unique $\mathcal{C}^\infty-$structure such that the inclusion map:

$$\iota_B^A : B \hookrightarrow A$$

is a $\mathcal{C}^\infty-$homomorphism.

We are going to denote the class of all $\mathcal{C}^\infty-$subrings of a given $\mathcal{C}^\infty-$ring (A, Φ) by $\mathrm{Sub}\,(A, \Phi)$.

Next we prove that the intersection of any family of $\mathcal{C}^\infty-$subrings of a given $\mathcal{C}^\infty-$ring is, again, a $\mathcal{C}^\infty-$subring.

Proposition 2.2. *Let* $\{(A_\alpha, \Phi_\alpha) | \alpha \in \Lambda\}$ *be a family of* $\mathcal{C}^\infty-$*subrings of* (A, Φ)*, so*

$$(\forall \alpha \in \Lambda)(\forall n \in \mathbb{N})(\forall f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))(\Phi_\alpha(f) = \Phi(f) \restriction_{{A_\alpha}^n} : {A_\alpha}^n \to A_\alpha)$$

We have that:

$$\left(\bigcap_{\alpha \in \Lambda} A_\alpha, \Phi'\right)$$

where:

$$\begin{array}{rccc} \Phi' : & \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n \in \mathbb{N}} \mathrm{Func}\ \left(\left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^n, \bigcap_{\alpha \in \Lambda} A_\alpha\right) \\ & (\mathbb{R}^n \xrightarrow{f} \mathbb{R}) & \mapsto & \Phi(f) \restriction_{\left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^n} : \left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^n \to \bigcap_{\alpha \in \Lambda} A_\alpha \end{array}$$

is a $\mathcal{C}^\infty-$*subring of* (A, Φ).

Proof. See **Proposition 1**, p. 9 of [9]. □

As an application of the previous result, we can define the $\mathcal{C}^\infty-$subring generated by a subset of the carrier of a $\mathcal{C}^\infty-$ring:

Definition 2.3. Let (A, Φ) be a $\mathcal{C}^\infty-$ring and $X \subseteq A$. The $\mathcal{C}^\infty-$subring of (A, Φ) generated by X is given by:

$$\langle X \rangle = \bigcap_{\substack{X \subseteq A_i \\ (A_i, \Phi_i) \subseteq (A, \Phi)}} (A_i, \Phi_i),$$

where $(A_i, \Phi_i) \subseteq (A, \Phi)$ means that (A_i, Φ_i) is a $\mathcal{C}^\infty-$subring of (A, Φ) together with the $\mathcal{C}^\infty-$structure given in **Proposition 2.2**.

We note that, given any $\mathcal{C}^\infty-$ring (A, Φ), the map of partially ordered sets given by:

$$\begin{array}{rccc} \sigma : & (\wp(A), \subseteq) & \to & (\,\mathsf{Sub}\,(A), \subseteq) \\ & X & \mapsto & \langle X \rangle \end{array}$$

satisfies the axioms of a closure operation.

2.2. Directed union of $\mathcal{C}^\infty-$rings

In general, given an arbitrary family $(A_\alpha, \Phi_\alpha)_{\alpha \in \Lambda}$ of $\mathcal{C}^\infty-$subrings of a given $\mathcal{C}^\infty-$ring (A, Φ), its union, $\bigcup_{\alpha \in \Lambda} A_\alpha$, together with $\Phi \upharpoonright_{\bigcup_{\alpha\in\Lambda} A_\alpha}$, is not necessarily a $\mathcal{C}^\infty-$subring of (A, Φ). However, there is an important case in which the union of a family of $\mathcal{C}^\infty-$subrings of a $\mathcal{C}^\infty-$ring (A, Φ) is, again, a $\mathcal{C}^\infty-$ring. This case is discussed in the following:

Proposition 2.4. *Let (A, Φ) be a $\mathcal{C}^\infty-$ring and let $\{(A_\alpha, \Phi_\alpha) | \alpha \in \Lambda\}$, $\Lambda \neq \varnothing$, be a directed family of $\mathcal{C}^\infty-$subrings of (A, Φ), that is, a family such that for every pair $(\alpha, \beta) \in \Lambda \times \Lambda$ there is some $\gamma \in \Lambda$ such that:*

$$A_\alpha \subseteq A_\gamma$$

and

$$A_\beta \subseteq A_\gamma$$

We have that:

$$\left(\bigcup_{\alpha \in \Lambda} A_\alpha, \Phi' \right)$$

where:

$$\begin{array}{rccc} \Phi' : & \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\ \left(\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^n, \bigcup_{\alpha\in\Lambda} A_\alpha \right) \\ & (\mathbb{R}^n \xrightarrow{f} \mathbb{R}) & \mapsto & \Phi(f) \upharpoonright_{\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^n} : \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^n \to \bigcup_{\alpha\in\Lambda} A_\alpha \end{array}$$

is a $\mathcal{C}^\infty-$subring of (A, Φ).

Proof. See **Proposition 2**, p. 10 of [9]. □

2.3. Products, $\mathcal{C}^\infty-$Congruences and Quotients

Next we describe the products in the category $\mathcal{C}^\infty$**Ring**, that is, products of arbitrary families of $\mathcal{C}^\infty-$rings.

Definition 2.5. Let $\{(A_\alpha, \Phi_\alpha) | \alpha \in \Lambda\}$ be a family of $\mathcal{C}^\infty-$rings. The **product** of this family is the pair:

$$\left(\prod_{\alpha \in \Lambda} A_\alpha, \Phi^{(\Lambda)}\right)$$

where Φ is given by:

$$\begin{array}{rccc} \Phi^{(\Lambda)}: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\left(\left(\prod_{\alpha\in\Lambda} A_\alpha\right)^n, \prod_{\alpha\in\Lambda} A_\alpha\right) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \begin{array}{rcl} \Phi^{(\Lambda)}(f) : \left(\prod_{\alpha\in\Lambda} A_\alpha\right)^n & \to & \prod_{\alpha\in\Lambda} A_\alpha \\ ((x^1_\alpha)_{\alpha\in\Lambda}, \cdots, (x^n_\alpha)_{\alpha\in\Lambda}) & \mapsto & (\Phi_\alpha(f)(x^1_\alpha, \cdots, x^n_\alpha))_\alpha \end{array} \end{array}$$

Remark 2.6. In particular, given a $\mathcal{C}^\infty-$ring (A, Φ), we have the product $\mathcal{C}^\infty-$ring:

$$(A \times A, \Phi^{(2)})$$

where:

$$\begin{array}{rccc} \Phi^{(2)}: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,((A\times A)^n, A\times A) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & (\Phi\times\Phi)(f) : (A\times A)^n \to A \times A \end{array}$$

and:

$$\begin{array}{rccc} \Phi^{(2)}(f): & (A\times A)^n & \to & A\times A \\ & ((x_1, y_1), \cdots, (x_n, y_n)) & \mapsto & (\Phi(f)(x_1, \cdots, x_n), \Phi(f)(y_1, \cdots, y_n)) \end{array}$$

We turn now to the definition of congruence relations in $\mathcal{C}^\infty-$rings. As we shall see later on, the congruences of a $\mathcal{C}^\infty-$rings will be classified by their ring-theoretic ideals.

Definition 2.7. Let (A, Φ) be a $\mathcal{C}^\infty-$ring. A $\mathcal{C}^\infty-$**congruence** is an equivalence relation $R \subseteq A \times A$ such that for every $n \in \mathbb{N}$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ we have:

$$(x_1, y_1), \cdots, (x_n, y_n) \in R \Rightarrow \Phi^{(2)}(f)((x_1, y_1), \cdots, (x_n, y_n)) \in R$$

In other words,a $\mathcal{C}^\infty-$congruence is an equivalence relations that preserves $\mathcal{C}^\infty-$function symbols.

A characterization of a $\mathcal{C}^\infty-$congruence can be given using the product $\mathcal{C}^\infty-$structure, as we see in the following:

Proposition 2.8. *Let (A,Φ) be a $\mathcal{C}^\infty-$ring and let $R \subseteq A \times A$ be an equivalence relation. Under these circumstances, R is a $\mathcal{C}^\infty-$congruence on (A,Φ) if, and only if, $(R, {\Phi^{(2)}}')$, where:*

$$\begin{array}{rccc} {\Phi^{(2)}}': & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(R^n, R) \\ & (\mathbb{R}^n \xrightarrow{f} \mathbb{R}) & \mapsto & \Phi^{(2)}(f)\restriction_{R^n}: R^n \to R \end{array}$$

is a $\mathcal{C}^\infty-$subring of $(A\times A, \Phi^{(2)})$, with the structure described in the ***Remark 2.6****.*

Proof. See **Proposition 3**, p. 13 of [9]. □

Remark 2.9. Given a $\mathcal{C}^\infty-$ring (A,Φ) and a $\mathcal{C}^\infty-$congruence $R \subseteq A \times A$, let:

$$(A/R) = \{\overline{a} | a \in A\}$$

be the quotient set. Given any $n \in \mathbb{N}$, $f \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ and $(\overline{a_1}, \cdots, \overline{a_n}) \in ((A/R))^n$ we define:

$$\begin{array}{rccc} \overline{\Phi}: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(((A/R))^n, (A/R)) \\ & (f: \mathbb{R}^n \overset{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \left(\overline{\Phi}(f): ((A/R))^n \to (A/R)\right) \end{array}$$

where:

$$\begin{array}{rccc} \overline{\Phi}(f): & ((A/R))^n & \to & (A/R) \\ & (\overline{a_1}, \cdots, \overline{a_n}) & \mapsto & \overline{\Phi(f)(a_1, \cdots, a_n)} \end{array}$$

Note that the interpretation above is indeed a function, that is, its value does not depend on any particular choice of the representing element. This means that given $(\overline{a_1}, \cdots, \overline{a_n}), (\overline{a_1'}, \cdots, \overline{a_n'}) \in ((A/R))^n$ such that $(a_1, a_1'), \cdots, (a_n, a_n') \in R$, we have:

$$\overline{\Phi}(f)(\overline{a_1}, \cdots, \overline{a_n}) = \overline{\Phi(f)(a_1, \cdots, a_n)}$$

and since R is a $\mathcal{C}^\infty-$congruence,

$$(a_1, a_1'), \cdots, (a_n, a_n') \in R \Rightarrow (\Phi(f)(a_1, \cdots, a_n), \Phi(f)(a_1', \cdots, a_n')) \in R$$

so:

$$\overline{\Phi}(\overline{a_1}, \cdots, \overline{a_n}) = \overline{\Phi(f)(a_1, \cdots, a_n)} = \overline{\Phi(f)(a_1', \cdots, a_n')} = \overline{\Phi}(f)(\overline{a_1'}, \cdots, \overline{a_n'})$$

The above construction leads directly to the following:

Definition 2.10. Let (A,Φ) be a $\mathcal{C}^\infty-$ring and let $R \subseteq A \times A$ be a $\mathcal{C}^\infty-$congruence. The **quotient** $\mathcal{C}^\infty-$**ring** of A by R is the ordered pair:

$$\left((A/R), \overline{\Phi}\right)$$

where:

$$(A/R) = \{\overline{a} | a \in A\}$$

and

$$\begin{array}{rccc} \overline{\Phi}: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \rightarrow & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\left(((A/R))^n, (A/R)\right) \\ & (f: \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\rightarrow} \mathbb{R}) & \mapsto & \left(\overline{\Phi}(f): ((A/R))^n \rightarrow (A/R)\right) \end{array}$$

where $\overline{\Phi}(f)$ is described in **Remark 2.9**.

The following result shows that the canonical quotient map is, again, a $\mathcal{C}^\infty-$homomorphism.

Proposition 2.11. *Let (A,Φ) be a $\mathcal{C}^\infty-$ring and let $R \subseteq A \times A$ be a $\mathcal{C}^\infty-$congruence. The function:*

$$\begin{array}{rccc} q: & (A,\Phi) & \rightarrow & \left((A/R), \overline{\Phi}\right) \\ & a & \mapsto & \overline{a} \end{array}$$

is a $\mathcal{C}^\infty-$homomorphism.

Proof. See **Proposition 4**, p. 15 of [9]. □

We remark that the structure given above is the unique $\mathcal{C}^\infty-$structure such that the quotient map is a $\mathcal{C}^\infty-$homomorphism.

Proposition 2.12. *Let (A,Φ) and (B,Ψ) be two $\mathcal{C}^\infty-$rings and let $\varphi : (A,\Phi) \rightarrow (B,\Psi)$ be a $\mathcal{C}^\infty-$homomorphism. The set:*

$$\ker(\varphi) = \{(a,a') \in A \times A | \varphi(a) = \varphi(a')\}$$

is a $\mathcal{C}^\infty-$congruence on (A,Φ).

Proof. See **Proposition 5** , p. 16 of [9]. □

Corollary 2.13. *Let (A,Φ) and (B,Ψ) be two $\mathcal{C}^\infty-$rings and let $\varphi : (A,\Phi) \rightarrow (B,\Psi)$ be a $\mathcal{C}^\infty-$homomorphism. Then $(\ker(\varphi), {\Phi^{(2)}}')$ is a $\mathcal{C}^\infty-$subring of $(A \times A, \Phi^{(2)})$.*

Proposition 2.14. *For every $\mathcal{C}^\infty-$congruence $R \subseteq A \times A$ in (A,Φ), there are some $\mathcal{C}^\infty-$ring (B,Ψ) and some $\mathcal{C}^\infty-$homomorphism $\varphi : (A,\Phi) \rightarrow (B,\Psi)$ such that $R = \ker(\varphi)$.*

Proof. It suffices to take $(B,\Psi) = \left(\frac{A}{R}, \overline{\Phi}\right)$ and $\varphi = q_R : (A,\Phi) \to \left(\frac{A}{R}, \overline{\Phi}\right)$. □

Theorem 2.15. (Fundamental Theorem of the $\mathcal{C}^\infty-$Homomorphism) *Let (A,Φ) be a $\mathcal{C}^\infty-$ring and $R \subseteq A \times A$ be a $\mathcal{C}^\infty-$congruence. For every $\mathcal{C}^\infty-$ring (B,Ψ) and for every $\mathcal{C}^\infty-$homomorphism $\varphi : (A,\Phi) \to (B,\Psi)$ such that $R \subseteq \ker(\varphi)$, that is, such that:*

$$(a, a') \in R \Rightarrow \varphi(a) = \varphi(a'),$$

there is a unique $\mathcal{C}^\infty-$homomorphism:

$$\widetilde{\varphi} : \left((A/R), \overline{\Phi}\right) \to (B,\Psi)$$

such that the following diagram commutes:

$$\begin{tikzcd}
(A,\Phi) \arrow[r, "\varphi"] \arrow[d, "q"'] & (B,\Psi) \\
\left((A/R),\overline{\Phi}\right) \arrow[ur, dashed, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

that is, such that $\widetilde{\varphi} \circ q = \varphi$, where $\overline{\Phi}$ is the canonical $\mathcal{C}^\infty-$structure induced on the quotient $\frac{A}{R}$

Proof. See **Theorem 3**, p. 17 of [9]. □

The following result is straightforward:

Proposition 2.16. *Let (A,Φ) and (B,Ψ) be two $\mathcal{C}^\infty-$rings and let $\varphi : (A,\Phi) \to (B,\Psi)$ be a $\mathcal{C}^\infty-$homomorphism. The ordered pair:*

$$(\varphi[A], \Psi')$$

where:

$$\begin{array}{rccc}
\Psi' : & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(\varphi[A]^n, \varphi[A]) \\
& (\mathbb{R}^n \xrightarrow{f} \mathbb{R}) & \mapsto & \Psi(f)\restriction_{\varphi[A]^n} : \varphi[A]^n \to \varphi[A]
\end{array}$$

*is a $\mathcal{C}^\infty-$subring of (B,Ψ), called **the homomorphic image of** A **by** φ.*

Corollary 2.17. *Let (A,Φ) and (B,Ψ) be two $\mathcal{C}^\infty-$rings and let $\varphi : (A,\Phi) \to (B,\Psi)$ be a $\mathcal{C}^\infty-$homomorphism. As we have noticed in **Proposition 2.16**, $(\varphi[A], \Psi')$ is a $\mathcal{C}^\infty-$subring of (B,Ψ).*

Under these circumstances, there is a unique $\mathcal{C}^\infty-$isomorphism:

$$\widetilde{\varphi} : \left(\frac{A}{\ker(\varphi)}, \overline{\Phi}\right) \to (\varphi[A], \Psi')$$

such that the following diagram commutes:

$$\begin{tikzcd}
(A,\Phi) \arrow[rr, "\varphi"] \arrow[d, "q"'] & & (\varphi[A],\Psi') \\
\left(\dfrac{A}{\ker(\varphi)},\overline{\Phi}\right) \arrow[urr, dashed, "\widetilde{\varphi}"'] & &
\end{tikzcd}$$

that is, such that $\widetilde{\varphi}\circ q = \varphi$, *where* $\overline{\Phi}$ *is the canonical* $\mathcal{C}^\infty-$*structure induced on the quotient* $\frac{A}{\ker(\varphi)}$

Proof. See **Corollary 2**, p. 19 of [9]. □

2.4. Directed Colimits of $\mathcal{C}^\infty-$Rings

The following result is going to be used to construct directed colimits of $\mathcal{C}^\infty-$rings.

Lemma 2.18. *Let* (A,Φ) *be a* $\mathcal{C}^\infty-$*ring. The ordered pair:*

$$(A\times\{\alpha\},\Phi\times \mathrm{id}_\alpha)$$

where:

$$\begin{array}{rcccc}
\Phi\times\mathrm{id}_\alpha: & \bigcup_{n\in\mathbb{N}}\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}) & \to & \multicolumn{2}{c}{\bigcup_{n\in\mathbb{N}}\mathrm{Func}\,((A\times\{\alpha\})^n, A\times\{\alpha\})} \\
 & (\mathbb{R}^n \xrightarrow{f} \mathbb{R}) & \mapsto & \Phi(f)\times\mathrm{id}_\alpha : (A\times\{\alpha\})^n & \to \quad A\times\{\alpha\} \\
 & & & ((a_1,\alpha),\cdots,(a_n,\alpha)) & \mapsto (\Phi(f)(a_1,\cdots,a_n),\alpha)
\end{array}$$

is a $\mathcal{C}^\infty-$*ring and:*

$$\begin{array}{rccc}
\pi_1: & A\times\{\alpha\} & \to & A \\
 & (a,\alpha) & \mapsto & a
\end{array}$$

is a $\mathcal{C}^\infty-$*isomorphism, that is:*

$$(A,\Phi)\cong_{\pi_1}(A\times\{\alpha\},\Phi\times\mathrm{id}_\alpha)$$

Proof. See **Lemma 1**, p. 20 of [9]. □

The proof of the following result describes the construction of directed colimits of directed families of $\mathcal{C}^\infty-$rings.

Theorem 2.19. *Let $(I, \leq)$ be a directed set and let $((A_\alpha, \Phi_\alpha), \mu_{\alpha\beta})_{\alpha,\beta \in I}$ be a directed system. There is an object (A, Φ) in $\mathcal{C}^\infty$**Ring** such that:*

$$(A, \Phi) \cong \varinjlim_{\alpha \in I} (A_\alpha, \Phi_\alpha)$$

Proof. See **Theorem 4**, p. 22 of [9]. □

Theorem 2.20. *Given any small category $\mathcal{J}$ and any diagram:*

$$\begin{array}{rccc} D: & \mathcal{J} & \rightarrow & \mathcal{C}^\infty\textbf{Ring} \\ & (\alpha \stackrel{h}{\rightarrow} \beta) & \mapsto & (A_\alpha, \Phi_\alpha) \stackrel{D(h)}{\rightarrow} (A_\beta, \Phi_\beta) \end{array}$$

there is a $\mathcal{C}^\infty-$ring (A, Φ) such that:

$$(A, \Phi) \cong \varprojlim_{\alpha \in I} D(\alpha)$$

Proof. See **Theorem 5**, p. 26 of [9]. □

Remark 2.21. Let $\Sigma = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ be a denumerable set of variables, so $F(\Sigma, X)$ will denote the algebra of terms of this language Σ. A class of **ordered pairs** will be simply a subset $S \subseteq F(\Sigma, X) \times F(\Sigma, X)$. In our case, these pairs are given by the axioms, so S consists of the following:

- For any $n \in \mathbb{R}$, $i \leq n$ and a (smooth) projection map $p_i : \mathbb{R}^n \to \mathbb{R}$ we have:

$$(p_i(x_1, \cdots, x_i, \cdots, x_n), x_i) \in S$$

- for every $f, g_1, \cdots, g_n \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ and $h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = h \circ (g_1, \cdots, g_n)$, we have

$$(h(g_1(x_1, \cdots, x_m), \cdots, g_n(x_1, \cdots, x_m)), f(x_1, \cdots, x_m)) \in S$$

Remark 2.22. The class of $\mathcal{C}^\infty-$rings is a model of an equational theory, thus it is a variety of algebras. However, if we were not given this information, noting that the category of $\mathcal{C}^\infty-$rings is closed under products, subalgebras and homomorphic images, the **HSP Birkhoff's Theorem** would lead us to the same conclusion, that is, that the class $\mathcal{C}^\infty$**Ring** is a variety of algebras, and by the previous remark, $\mathcal{C}^\infty$**Ring** $= V(S)$, the variety of algebras defined by S.

In particular, we have some classical results. We list some of them:

- for every set X there is a free $\mathcal{C}^\infty-$ring determined by X;

- any $\mathcal{C}^\infty-$ring is a homomorphic image of some free $\mathcal{C}^\infty-$ring;
- a $\mathcal{C}^\infty-$homomorphism is monic if, and only if, it is an injective map;
- any indexed set of $\mathcal{C}^\infty-$rings, $\{(A_\alpha, \Phi_\alpha) | \alpha \in I\}$, has a coproduct in $\mathcal{C}^\infty$**Ring**.

We end this section by stating a result which says that the (variety) of all $\mathcal{C}^\infty-$rings is a reflective subcategory of $\mathcal{C}^\infty$**Str**.

Theorem 2.23. *The inclusion functor $\iota : \mathcal{C}^\infty\textbf{Ring} \hookrightarrow \mathcal{C}^\infty\textbf{Str}$ has a left adjoint $L : \mathcal{C}^\infty\textbf{Str} \to \mathcal{C}^\infty\textbf{Ring}$: given by $M \mapsto M/\theta_M$ where θ_M is the least $\mathcal{C}^\infty$-congruence of M such that $M/\theta_M \in \mathrm{Obj}(\mathcal{C}^\infty\textbf{Ring})$. Moreover, the unit of the adjunction $L \dashv \iota$ has components $(q_M)_{M \in \mathrm{Obj}(\mathcal{C}^\infty\textbf{Str})}$, where $q_M : M \twoheadrightarrow M/\theta_M$ is the quotient homomorphism.*

Proof. See **Theorem 6**, p. 27 of [9]. □

3. Free $\mathcal{C}^\infty-$Rings

Our definition of $\mathcal{C}^\infty-$ring yields a forgetful functor:

$$\begin{array}{cccc} U: & \mathcal{C}^\infty\textbf{Ring} & \to & \textbf{Set} \\ & (A,\Phi) & \mapsto & A \\ & ((A,\Phi) \stackrel{\varphi}{\to} (B,\Psi)) & \mapsto & (A \stackrel{U(\varphi)}{\to} B) \end{array}$$

In fact, as we are going to see later, this functor has a left adjoint, the "free $\mathcal{C}^\infty-$ring", that we shall denote by $L : \textbf{Set} \to \mathcal{C}^\infty\textbf{Ring}$. Before we do it, we need the following:

Remark 3.1. Given any $m \in \mathbb{N}$, we note that the set $\mathcal{C}^\infty(\mathbb{R}^n)$ may be endowed with a $\mathcal{C}^\infty-$structure:

$$\begin{array}{cccc} \Omega: & \bigcup_{n\in\mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(\mathcal{C}^\infty(\mathbb{R}^m)^n, \mathcal{C}^\infty(\mathbb{R}^m)) \\ & (\mathbb{R}^n \stackrel{f}{\to} \mathbb{R}) & \mapsto & \Omega(f) = f \circ - : \begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^m)^n & \to & \mathcal{C}^\infty(\mathbb{R}^m) \\ (h_1, \cdots, h_n) & \mapsto & f \circ (h_1, \cdots, h_n) \end{array} \end{array}$$

so it is easy to see that it can be made into a $\mathcal{C}^\infty-$ring $(\mathcal{C}^\infty(\mathbb{R}^m), \Omega)$. From now, when dealing with this"canonical" $\mathcal{C}^\infty-$structure, we shall omit the symbol Ω, writing $\mathcal{C}^\infty(\mathbb{R}^m)$ instead of $(\mathcal{C}^\infty(\mathbb{R}^m), \Omega)$.

We have the following construction of finitely generated free $\mathcal{C}^\infty-$rings:

Proposition 3.2. *Let $U : \mathcal{C}^\infty\textbf{Ring} \to \textbf{Set}$, $(A, \Phi) \mapsto A$, be the forgetful functor. The pair $(\jmath_n, (\mathcal{C}^\infty(\mathbb{R}^n), \Omega))$, where:*

$$\begin{array}{cccc} \jmath_n: & \{1, \cdots, n\} & \to & U(\mathcal{C}^\infty(\mathbb{R}^n), \Omega) \\ & i & \mapsto & \pi_i : \mathbb{R}^n \to \mathbb{R} \end{array},$$

is the free $\mathcal{C}^\infty$-ring with n generators, which are the projections:

$$\begin{array}{rccc} \pi_i : & \mathbb{R}^n & \to & \mathbb{R} \\ & (x_1, \cdots, x_i, \cdots, x_n) & \mapsto & x_i \end{array}$$

Proof. See **Proposition 1.1** of [1]. □

As a consequence of the above result, for any finite set X, $(\jmath_X, (\mathcal{C}^\infty(\mathbb{R}^X), \Phi_X))$ is the free $\mathcal{C}^\infty-$ring defined by X (see **Corollary 3**, p. 42 of [9]). One extends the definition for free $\mathcal{C}^\infty-$rings generated by infinite sets by using the colimit of the finitely generated free ones. Thus, given an (infinite) set E, one decomposes it as the union of its finite subsets:

$$E = \bigcup_{E' \subseteq_{\mathrm{fin}} E} E'$$

and defines:

$$\mathcal{C}^\infty(\mathbb{R}^E) = \varinjlim_{E' \subseteq_{\mathrm{fin}} E} \mathcal{C}^\infty(\mathbb{R}^{E'}).$$

Formally we have:

Proposition 3.3. *Let E be any set. The pair $(\jmath_E, (\mathcal{C}^\infty(\mathbb{R}^E), \Phi_E))$ (where $\jmath_E : E \to U(\mathcal{C}^\infty(\mathbb{R}^E), \Phi_E)$ is given (uniquely) by the universal property of the colimit $\mathcal{C}^\infty(\mathbb{R}^E)$) is the free $\mathcal{C}^\infty-$ring determined by E.*

Proof. See **Proposition 10**, p. 44 of [9]. □

The underlying set $\mathcal{C}^\infty(\mathbb{R}^E)$ can be realized as a subset of $\mathrm{Func}\,(\mathbb{R}^E, \mathbb{R})$, namely as the subset of all functions from $\mathbb{R}^E$ to $\mathbb{R}$ which depend smoothly on finitely many coordinates (cf. **Section 3** of [9]).

In the following proposition, we present a description of a left adjoint to the forgetful functor $U : \mathcal{C}^\infty\mathbf{Ring} \to \mathbf{Set}$.

Proposition 3.4. *The functions:*

$$\begin{array}{rccc} L_0 : & \mathrm{Obj}\,(\boldsymbol{Set}) & \to & \mathrm{Obj}\,(\mathcal{C}^\infty\boldsymbol{Ring}) \\ & X & \mapsto & (\mathcal{C}^\infty\,(\mathbb{R}^X), \Phi_X) \end{array}$$

and

$$\begin{array}{rccc} L_1 : & \mathrm{Mor}\,(\boldsymbol{Set}) & \to & \mathrm{Mor}\,(\mathcal{C}^\infty\boldsymbol{Ring}) \\ & (X \stackrel{f}{\to} Y) & \mapsto & (\mathcal{C}^\infty(\mathbb{R}^X), \Phi_X) \stackrel{\widetilde{f}}{\to} (\mathcal{C}^\infty(\mathbb{R}^Y), \Phi_Y) \end{array}$$

where $\widetilde{f} : L_0(X) \to L_0(Y)$ is the unique $\mathcal{C}^\infty-$homomorphism given by the universal property of the free $\mathcal{C}^\infty-$ring $\jmath_X : X \to \mathcal{C}^\infty(\mathbb{R}^X)$: given the function $\jmath_Y \circ f : X \to U(\mathcal{C}^\infty(\mathbb{R}^Y))$, there is a unique $\mathcal{C}^\infty-$homomorphism such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\jmath_X"] \arrow[dr, "\jmath_Y \circ f"'] & U(\mathcal{C}^\infty(\mathbb{R}^X)) \arrow[d, dashed, "U(\widetilde{f})"] \\ & U(\mathcal{C}^\infty(\mathbb{R}^Y)) \end{tikzcd}$$

*define a functor L : **Set** $\to \mathcal{C}^\infty$**Ring** which is left adjoint to the forgetful functor U : $\mathcal{C}^\infty$**Ring** $\to$ **Set**.*

Proof. See **Proposition 11**, p. 47 of [9]. □

3.1. Relations and Generators

Let (A, Φ) be a $\mathcal{C}^\infty-$ring and let $X \subseteq A$. Given the inclusion map $\iota_X^A : X \hookrightarrow A$, there is a unique $\mathcal{C}^\infty-$homomorphism $\widetilde{\iota_X^A} : (L(X), \Phi_X) \to (A, \Phi)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\iota_X^A"'] & U(L(X)) \arrow[d, "U(\widetilde{\iota_X^A})"] \\ & U(A, \Phi) \end{tikzcd} \tag{1}$$

We claim that if $\langle X \rangle = A$, then $\widetilde{\iota_X^A}$ is surjective.

Indeed,

$$X = \iota_X^A[X] = \mathrm{im}(\iota_X^A)$$

and since $U(\widetilde{\iota_X^A}) \circ \eta_X = \iota_X^A$, we have:

$$\mathrm{im}(\iota_X^A) = \mathrm{im}(U(\widetilde{\iota_X^A}) \circ \eta_X).$$

Since the diagram (1) commutes, on the other hand,

$$\mathrm{im}(U(\widetilde{\iota_X^A}) \circ \eta_X) \subseteq \mathrm{im}\,(U(\widetilde{\iota_X^A}))$$

thus $X \subseteq \mathrm{im}\,(U(\widetilde{\iota_X^A}))$ and $\langle X \rangle \subseteq \langle \mathrm{im}\,(U(\widetilde{\iota_X^A})) \rangle = \widetilde{\iota_X^A}[A]$.

Since X generates A and $\langle \mathrm{im}\,(U(\widetilde{\iota_X^A})) \rangle = \mathrm{im}\,(\widetilde{\iota_X^A})$ is a $\mathcal{C}^\infty-$subring of (A, Φ), it follows that:

$$\langle X \rangle = A \subseteq \mathrm{im}(\widetilde{\iota_X^A}) \subseteq A,$$

so $\mathrm{im}\,(\widetilde{\iota_X^A}) = A$ and ι_X^A is surjective.

In particular, taking $X = A$ yields $\iota_A^A = \mathrm{id}_A$, and since $\varepsilon_A = \phi_{A,(A,\Phi)}(\mathrm{id}_A) = \widetilde{\mathrm{id}_{(A,\Phi)}}$, we have:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & U(L(A)) \arrow[d, "U(\varepsilon_A)"] \\ & U(A,\Phi) \end{tikzcd}$$

so $\mathrm{im}\,(\varepsilon_A) = (A,\Phi)$, and ε_A is surjective. Now, given any $\mathcal{C}^\infty-$ring (A,Φ) we have the surjective morphism:

$$\varepsilon_A : L(U(A,\Phi)) \twoheadrightarrow (A,\Phi).$$

We have seen that since ε_A is a $\mathcal{C}^\infty-$homomorphism, $\ker(\varepsilon_A)$ is a $\mathcal{C}^\infty-$congruence. By the **Fundamental Theorem of the $\mathcal{C}^\infty-$Isomorphism** we have:

$$(A,\Phi) \cong \frac{L(A)}{\ker(\varepsilon_A)}$$

As an application, we register the following (very useful) result:

Theorem 3.5. *Let (A,Φ) be any $\mathcal{C}^\infty-$ring. There is a directed system of $\mathcal{C}^\infty-$rings, $((A_i,\Phi_i),\alpha_{ij})$, where each (A_i,Φ_i) is a finitely generated $\mathcal{C}^\infty-$ring and each $\alpha_{ij} : (A_i,\Phi_i) \to (A_j,\Phi_j)$ is a monomorphism such that:*

$$(A,\Phi) \cong \varinjlim (A_i,\Phi_i)$$

Proof. See **Theorem 8**, p. 54 of [9]. □

4. Other Constructions

In this section we describe further results and constructions involving $\mathcal{C}^\infty-$rings.

4.1. Ring-Theoretic Ideals and $\mathcal{C}^\infty-$Congruences

Our next goal is to classify the congruences of any $\mathcal{C}^\infty-$ring. We shall see that they are classified by their ring-theoretic ideals.

Since each $\mathcal{C}^\infty-$ring has an underlying (commutative, unital) ring, it is easy to see that the following result holds:

Proposition 4.1. *Given any $\mathcal{C}^\infty-$ring (A,Φ), let* $\mathrm{Cong}\,(A,\Phi)$ *denote the set of all the $\mathcal{C}^\infty-$congruences in A and let $\mathfrak{I}(A,\Phi)$ denote the set of all ideals of A. The following map is well-defined:*

$$\begin{array}{rccc} \psi_A: & \mathrm{Cong}\,(A,\Phi) & \rightarrow & \mathfrak{I}(A,\Phi) \\ & R & \mapsto & \{x \in A | (x,0) \in R\} \end{array}$$

In fact, we will show that, for each $\mathcal{C}^\infty-$ring (A,Φ), the map ψ_A is a bijection whose inverse map is given by

$$\begin{array}{rccc} \varphi_A: & \mathfrak{I}(A,\Phi) & \rightarrow & \mathrm{Cong}\,(A,\Phi) \\ & I & \mapsto & \{(x,y) \in A \times A | x-y \in I\} \end{array}$$

The point here is to show that the map φ_A is well-defined (i.e., if $I \subseteq A$ is an ideal, then $\varphi_A(I)$ is a $\mathcal{C}^\infty$-congruence). This is achieved through a sequence of steps[2]:

(I) We show that the statement holds for any <u>free finitely generated</u> $\mathcal{C}^\infty$-ring A ($A \cong \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$);

(II) We show that the statement holds for any <u>finitely generated</u> $\mathcal{C}^\infty$-ring B ($B \cong \dfrac{\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})}{\Theta}$, for some $\Theta \in \mathrm{Cong}(\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}))$);

(III) Finally, we show that the statement holds for <u>any</u> $\mathcal{C}^\infty$-ring C, by using **Theorem 3.5** ($C \cong \varinjlim_{i\in I} B_i$, for some directed diagram of finitely generated $\mathcal{C}^\infty$-rings).

We begin with the free finitely generated case.

An interesting result, which is a consequence of **Hadamard's Lemma**, **Theorem 4.2**, is the description of the ideals of free finitely generated $\mathcal{C}^\infty-$rings in terms of $\mathcal{C}^\infty-$congruences.

For the reader's benefit, we state the following result:

Theorem 4.2. **(Hadamard's Lemma)***For every smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there are smooth functions $g_1,\cdots,g_n \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ such that $\forall (x_1,\cdots,x_n),(y_1,\cdots,y_n) \in \mathbb{R}^n$:*

$$f(x_1,\cdots,x_n) - f(y_1,\cdots,y_n) = \sum_{i=1}^{n}(x_i-y_i)\cdot g_i(x_1,\cdots,x_n,y_1,\cdots,y_n)$$

Proof. See **Theorem 7**, p. 51 of [9]. □

[2]This procedure works in many situations!

Corollary 4.3. *Given a free finitely generated $\mathcal{C}^\infty-$ring (A,Φ), considering the forgetful functor given in **Remark 1.8**, $\widetilde{U} : \mathcal{C}^\infty$**Ring** $\to$ **CRing**, we have that if I is a subset of A that is an ideal (in the ordinary ring-theoretic sense) in $\widetilde{U}(A,\Phi)$, then $\widehat{I} = \{(a,b) \in A \times A | a-b \in I\}$ is a $\mathcal{C}^\infty-$congruence in A.*

Proof. See **Proposition 12**, p. 51 of [9]. □

The following lemma is a well-known result of Universal Algebra applied to $\mathcal{C}^\infty-$rings:

Lemma 4.4. *Let (A,Φ) be a $\mathcal{C}^\infty-$ring and let $R \in \mathrm{Cong}\,(A,\Phi)$. Given the quotient $\mathcal{C}^\infty-$homomorphism:*

$$\begin{array}{cccc} q_R: & (A,\Phi) & \to & \left((A/R),\overline{\Phi}\right) \\ & x & \mapsto & x+R \end{array}$$

we have the bijection:

$$\begin{array}{cccc} (q_R)_*: & \{S \in \mathrm{Cong}\,(A,\Phi) | R \subseteq S\} & \to & \mathrm{Cong}\left((A/R),\overline{\Phi}\right) \\ & S & \mapsto & \{(q_R(s),q_R(t)) \in \frac{A}{R} \times \frac{A}{R} | (s,t) \in S\} \end{array}$$

whose inverse is given by:

$$\begin{array}{cccc} (q_R)^*: & \mathrm{Cong}\left((A/R),\overline{\Phi}\right) & \to & \{S \in \mathrm{Cong}(A,\Phi) | R \subseteq S\} \\ & S' & \mapsto & (q_R \times q_R)^{\dashv}[S'] \end{array}$$

As a consequence of the above lemma, we have:

Lemma 4.5. *Let (A,Φ) be a $\mathcal{C}^\infty-$ring and $R \in \mathrm{Cong}\,(A,\Phi)$, and suppose that $\psi_A : \mathrm{Cong}\,(A,\Phi) \to \mathfrak{I}(A,\Phi)$ is a bijection with an inverse, $\varphi_A : \mathfrak{I}(A,\Phi) \to \mathrm{Cong}\,(A,\Phi)$. Under those circumstances, the quotient $\mathcal{C}^\infty-$homomorphism:*

$$q_R : (A,\Phi) \to \left((A/R),\overline{\Phi}\right)$$

induces a pair of inverse bijections:

$$\begin{array}{cccc} (q_R)_+: & \mathfrak{I}\left((A/R),\overline{\Phi}\right) & \to & \{I' \in \mathfrak{I}(A,\Phi) | \psi_A(R) \subseteq I'\} \\ & J & \mapsto & q_R[J] \end{array}$$

and

$$\begin{array}{rccc} (p_R)^{-}: & \{I' \in \mathfrak{I}(A,\Phi) | \psi_A(R) \subseteq I'\} & \to & \mathfrak{I}\left((A/R), \overline{\Phi}\right) \\ & J' & \mapsto & (q_R)^{\dashv}[J'] \end{array}$$

Proof. See **Lemma 4**, p. 64 of [9]. $\square$

The following result tells us that the ideals of the finitely generated $\mathcal{C}^{\infty}-$rings classify its congruences.

Proposition 4.6. *Given any finitely generated $\mathcal{C}^{\infty}-$ring (A,Φ). Then the following maps are well-defined and provide a pair of inverse bijections:*

$$\begin{array}{rccc} \psi_A: & \mathrm{Cong}\,(A,\Phi) & \to & \mathfrak{I}(A,\Phi) \\ & R & \mapsto & \{x \in A | (x,0) \in R\} \end{array}$$

$$\begin{array}{rccc} \varphi_A: & \mathfrak{I}(A,\Phi) & \to & \mathrm{Cong}\,(A,\Phi) \\ & I & \mapsto & \{(x,y) \in A \times A | x - y \in I\} \end{array}$$

Proof. See **Proposition 13**, p. 52 of [9]. $\square$

For any $\mathcal{C}^{\infty}-$ring (A,Φ) we have the function:

$$\begin{array}{rccc} \psi_A: & \mathrm{Cong}\,(A,\Phi) & \to & \mathfrak{I}(A,\Phi) \\ & R & \mapsto & \{a \in A | (a, \Phi(0)) \in R\} \end{array}$$

and whenever (A,Φ) is a finitely generated $\mathcal{C}^{\infty}-$ring, we have seen in **Proposition 4.6** that ψ_A has φ_A as inverse - which is a consequence of **Hadammard's Lemma**.

In order to show that ψ_A is a bijection for any $\mathcal{C}^{\infty}-$ring (A,Φ), first we decompose it as a directed colimit of its finitely generated $\mathcal{C}^{\infty}-$subrings (cf. **Theorem 3.5**):

$$(A,\Phi) \cong \varinjlim_{(A_i,\Phi_i) \subseteq_{\mathrm{f.g.}} (A,\Phi)} (A_i, \Phi \upharpoonright_{A_i})$$

and then we use **Proposition 4.6** to obtain a bijection:

$$\widetilde{\varphi}: \varprojlim_{(A_i,\Phi_i) \subseteq_{\mathrm{f.g.}} (A,\Phi)} \mathfrak{I}(A_i,\Phi_i) \to \varprojlim_{(A_i,\Phi_i) \subseteq_{\mathrm{f.g.}} (A,\Phi)} \mathrm{Cong}\,(A_i,\Phi_i)$$

such that for every $(A_i,\Phi_i) \subseteq_{\mathrm{f.g.}} (A,\Phi)$ the following diagram commutes:

$$\begin{tikzcd}
\varprojlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)} \mathfrak{I}(A_i,\Phi_i) \arrow[r, "\widetilde{\varphi}"] \arrow[d] & \varprojlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)} \mathrm{Cong}\,(A_i,\Phi_i) \arrow[d] \\
\mathfrak{I}(A_i,\Phi_i) \arrow[r, "\varphi_{A_i}"'] & \mathrm{Cong}(A_i,\Phi_i)
\end{tikzcd}$$

where the vertical downward arrows are the canonical arrows of the projective limit.

Finally we show that there is a bijective correspondence, α, between

$$\varprojlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)} \mathfrak{I}(A_i,\Phi_i)$$

and

$$\mathfrak{I}\left(\varinjlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)} (A_i,\Phi_i)\right)$$

and a bijective correspondence, β, between $\varprojlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)} \mathrm{Cong}\,(A_i,\Phi_i)$ and $\mathrm{Cong}\left(\varinjlim_{(A_i,\Phi_i)\subseteq_{\mathrm{f.g.}}(A,\Phi)}(A_i,\Phi_i)\right)$. In fact, the bijections $\alpha, \beta, \widetilde{\varphi}$ and ψ are complete lattices isomorphisms.

By composing these bijections we prove that the congruences of $(\mathcal{C}^\infty(\mathbb{R}^E), \Phi_E)$ are classified by the ring-theoretic ideals of $(\mathcal{C}^\infty(\mathbb{R}^E), \Phi_E)$.

As a consequence of this latter result, we have several examples of $\mathcal{C}^\infty-$rings.

Example 4.7. Given any $\mathcal{C}^\infty-$manifold M, the ring $\mathcal{C}^\infty(M)$ is a finitely presented $\mathcal{C}^\infty-$ring (**Theorem 2.3** of [1]). This is true because one can embed M in some $\mathbb{R}^n$, find an $\varepsilon-$neighbourhood $U \supset M$ and a retraction $r : U \to M$, so $\mathcal{C}^\infty(M)$ is a retract of $\mathcal{C}^\infty(U)$. (cf. p. 25 of [1]).

Example 4.8. As pointed out by I. Moerdijk and G. Reyes in [1], any Weil algebra (i.e., a local ring with an $\mathbb{R}-$algebra structure which, regarded as an $\mathbb{R}-$vector space is finite dimensional) is a $\mathcal{C}^\infty-$ring. Thus, rings of (Ehresmann) jets - *qua* Weil algebras - are also $\mathcal{C}^\infty-$rings.

Example 4.9. The ring of dual numbers, $\mathbb{R}[\varepsilon] = \frac{\mathbb{R}[X]}{\langle X^2\rangle}$ is a $\mathcal{C}^\infty-$ring, since by **Borel's Theorem** (see **Theorem 1.3** of [1]) we have:

$$\mathbb{R}[\varepsilon] \cong \frac{\mathcal{C}^\infty(\mathbb{R})}{\langle x^2\rangle}$$

Remark 4.10. Let $((A_i,\Phi_i)_{i\in I}, \alpha_{ij} : (A_i,\Phi_i) \to (A_j,\Phi_j))$ be an inductive directed system of $\mathcal{C}^\infty-$rings, and let $(J_\ell)_{\ell\in I} \in \varprojlim_{\ell\in I} \mathfrak{I}(A_\ell,\Phi_\ell)$.

For every $i \in I$, we have the maps:

$$\begin{array}{rccl} {\alpha_i}^* : & \mathfrak{I}\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) & \to & \mathfrak{I}(A_i, \Phi_i) \\ & J & \mapsto & \alpha_i^{\dashv}[J] \end{array}$$

$$\begin{array}{rccl} \widehat{\alpha} : & \mathfrak{I}\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) & \to & \mathfrak{I}(A_i, \Phi_i) \\ & J & \mapsto & (\alpha_i^{\dashv}[J])_{i \in I} \end{array}$$

and the following limit diagram:

$$\begin{tikzcd} & \varprojlim_{i \in I} \mathfrak{I}(A_i, \Phi_i) \arrow[dl, "\alpha_i^*"'] \arrow[dr, "\alpha_j^*"] & \\ \mathfrak{I}(A_i, \Phi_i) & & \mathfrak{I}(A_j, \Phi_j) \arrow[ll, "\alpha_{ij}^*"] \end{tikzcd},$$

where:

$$\begin{array}{rccl} {\alpha_{ij}}^* : & \mathfrak{I}(A_j, \Phi_j) & \to & \mathfrak{I}(A_i, \Phi_i) \\ & J_j & \mapsto & {\alpha_{ij}}^{\dashv}[J_j] \end{array}$$

We note, first, that α maps ideals of $\varinjlim_{i \in I}(A_i, \Phi_i)$ to an element of $\varprojlim_{i \in I} \mathfrak{I}(A_i, \Phi_i)$.

In fact, given any ideal $J \in \mathfrak{I}\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right)$, since for every $i \in I$, α_i is a $\mathcal{C}^\infty-$homomorphism, it follows that for every $i \in I$, $\alpha_i^*(J) = {\alpha_i}^{\dashv}[J]$ is an ideal of (A_i, Φ_i), so $\alpha(J) = (\alpha_i^*(J))_{i \in I} \in \prod_{i \in I} \mathfrak{I}(A_i, \Phi_i)$. Moreover, the family $(\alpha_i^*(J))_{i \in I}$ is compatible, since:

$$(\forall i \in I)(\forall j \in I)(i \preceq j)(\alpha_j \circ \alpha_{ij} = \alpha_i \Rightarrow \alpha_i^* = {\alpha_{ij}}^* \circ \alpha_j^*)$$

and

$$(\forall i \in I)(\forall j \in I)(i \preceq j)({\alpha_i}^*(J) = ({\alpha_{ij}}^* \circ \alpha_j^*)(J) = {\alpha_{ij}}^*(\alpha_j^*(J))$$

so

$$\alpha(J) = (\alpha_i^*(J))_{i \in I} \in \varprojlim_{i \in I} \mathfrak{I}(A_i, \Phi_i).$$

Proposition 4.11. *Let $(I, \preceq)$ be a directed partially ordered set and*

$$\{(A_i, \Phi_i), \alpha_{ij} : (A_i, \Phi_i) \to (A_j, \Phi_j)\}_{i,j \in I}$$

be a directed inductive system of $\mathcal{C}^\infty-$rings and $\mathcal{C}^\infty-$homomorphisms. For every $i \in I$, *we have the map:*

$$\begin{array}{rrcl} {\alpha_i}^* : & \Im\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) & \to & \Im(A_i, \Phi_i) \\ & J & \mapsto & \alpha_i^{\dashv}[J] \end{array}$$

By the universal property of $\varprojlim_{i\in I} \Im(A_i, \Phi_i)$, *there is a unique* $\alpha : \Im\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) \to \varprojlim_{i\in I} \Im(A_i, \Phi_i)$ *such that for every* $i \in I$ *the following diagram commutes:*

$$\begin{tikzcd} \Im\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) \arrow[d, "\exists!\alpha"'] \arrow[ddr, bend left, "\alpha_i^*"] & \\ \varprojlim_{i\in I} \Im(A_i, \Phi_i) \arrow[dr, "\pi_i\restriction_{\varprojlim_{i\in I} \Im(A_i,\Phi_i)}"'] & \\ & \Im(A_i, \Phi_i) \end{tikzcd}$$

that is, such that ${\alpha_i}^* = \pi_i \restriction_{\varprojlim_{i\in I} \Im(A_i,\Phi_i)} \circ \alpha$.

We have, thus:

$$\begin{array}{rrcl} \alpha : & \Im\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) & \to & \varprojlim_{i\in I} \Im(A_i, \Phi_i) \\ & J & \mapsto & (\alpha_i^*(J))_{i\in I} \end{array}$$

For any $(J_i)_{i\in I} \in \varprojlim_{i\in I} \Im(A_i, \Phi_i)$, $\bigcup_{i\in I} \alpha_i[J_i]$ *is an ideal of* $\varinjlim_{i\in I}(A_i, \Phi_i)$ *and the map:*

$$\begin{array}{rrcl} \alpha' : & \varprojlim_{i\in I} \Im(A_i, \Phi_i) & \to & \Im\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) \\ & (J_i)_{i\in I} & \mapsto & \bigcup_{i\in I} \alpha_i[J_i] \end{array}$$

is an inverse for α, *so* α *is a bijection.*

Proof. See **Proposition 14**, p. 58 of [9]. $\square$

The proof of the following result is similar to the proof of the above proposition:

Proposition 4.12. *Let* $(I, \preceq)$ *be a directed partially ordered set and* $\{(A_i, \Phi_i), \alpha_{ij} : (A_i, \Phi_i) \to (A_j, \Phi_j)\}_{i,j\in I}$ *be a directed inductive system of* $\mathcal{C}^\infty-$*rings and* $\mathcal{C}^\infty-$ *-homomorphisms. The following function is a bijection:*

$$\begin{array}{rrcl} \beta : & \mathrm{Cong}\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) & \to & \varprojlim_{i\in I} \mathrm{Cong}(A_i, \Phi_i) \\ & R & \mapsto & ((\alpha_i \times \alpha_i)^{\dashv}(R))_{i\in I} \end{array}$$

whose inverse is given by:

$$\begin{array}{rccc} \beta' : & \varprojlim_{i \in I} \mathrm{Cong}(A_i, \Phi_i) & \to & \mathrm{Cong}\left(\varinjlim_{i \in I}(A_i, \Phi_i)\right) \\ & (R_i)_{i \in I} & \mapsto & \varinjlim_{i \in I} R_i \end{array}$$

The following result extends **Proposition 4.6** in the sense that it shows us, with details, that the congruences of any free $\mathcal{C}^{\infty}-$ring are classified by their ring-theoretic ideals (in the finitely generated case it follows from Hadamard's lemma, and this case is used here).

Lemma 4.13. *The congruences of $(\mathcal{C}^{\infty}(\mathbb{R}^E), \Phi_E)$, the free $\mathcal{C}^{\infty}-$ring determined by the set E, are classified by their ring-theoretic ideals.*

Proof. See **Lemma 2**, p. 63 of [9]. □

Proposition 4.14. *Let (A, Φ) and (B, Ψ) be two $\mathcal{C}^{\infty}-$rings and let $h : (A, \Phi) \to (B, \Psi)$ be a surjective $\mathcal{C}^{\infty}-$homomorphism. The following functions are bijections:*

$$\begin{array}{rccc} h^* : & \mathrm{Cong}\,(B, \Psi) & \to & \{S \in \mathrm{Cong}\,(A, \Phi) \mid \ker(h) \subseteq S\} \\ & R & \mapsto & (h \times h)^{\dashv}[R] \end{array}$$

$$\begin{array}{rccc} h^- : & \mathfrak{I}(A, \Phi) & \to & \{I' \in \mathfrak{I}(B, \Psi) \mid \varphi_A(\ker(h)) \subseteq I'\} \\ & J & \mapsto & h^{\dashv}[J] \end{array}$$

Proof. See **Proposition 16**, p. 65 of [9]. □

The following theorem, subsumes the previous results and gives us a description of $\mathcal{C}^{\infty}-$rings via generators and relations.

Theorem 4.15. *Let (A, Φ) be any $\mathcal{C}^{\infty}-$ring. The $\mathcal{C}^{\infty}-$congruences of (A, Φ) are classified by the ring-theoretic ideals of (A, Φ).*

Proof. See **Proposition 17**, p. 65 of [9]. □

Given any $\mathcal{C}^{\infty}-$ring (A, Φ), there is a ring-theoretical ideal $I = \psi_A(\ker(\varepsilon_A))$ such that:

$$(A, \Phi) \cong \left(\frac{\mathcal{C}^{\infty}(\mathbb{R}^A)}{I}, \overline{\Phi}\right),$$

that is, every $\mathcal{C}^{\infty}-$ring is the quotient of a free $\mathcal{C}^{\infty}-$ring by some of its ring-theoretic ideals. We say that any $\mathcal{C}^{\infty}-$ring is given by generators and relations.

Remark 4.16. Let (A, Φ) be a $\mathcal{C}^{\infty}-$ring. The set $\mathrm{Cong}\,(A, \Phi)$ is partially ordered by inclusion. Also, given $\{R_i | i \in I\} \subseteq \mathrm{Cong}\,(A, \Phi)$, we have:

$$\bigcap_{i \in I} R_i \in \mathrm{Cong}\,(A, \Phi),$$

so we can define:

$$\begin{array}{rccc} \bigwedge : & \mathcal{P}(\mathrm{Cong}(A, \Phi)) & \rightarrow & \mathrm{Cong} \\ & \{R_i \mid i \in I\} & \mapsto & \bigcap_{i \in I}\{R_i \mid i \in I\} \end{array}.$$

Also, given $\{R_i \mid i \in I\} \subseteq \mathrm{Cong}\,(A, \Phi)$, we define:

$$\begin{array}{rccc} \bigvee : & \mathcal{P}(\mathrm{Cong}(A, \Phi)) & \rightarrow & \mathrm{Cong} \\ & \{R_i \mid i \in I\} & \mapsto & \bigcap\{R \in \mathrm{Cong}(A, \Phi) \mid \bigcup_{i \in I}\{R_i \mid i \in I\} \subseteq R\} \end{array},$$

so $(\mathrm{Cong}\,(A, \Phi), \bigwedge, \bigvee)$ is a complete lattice.

Note that $\mathfrak{I}(A, \Phi)$, partially ordered by inclusion, also has a structure of complete lattice, since it the set of ring-theoretic ideals of $(A, \Phi(+), \Phi(\cdot), \Phi(-), \Phi(0), \Phi(1))$.

We have constructed, in **Theorem 4.15**, a bijection:

$$\varphi_{(A,\Phi)} : \mathrm{Cong}\,(A, \Phi) \rightarrow \mathfrak{I}(A, \Phi).$$

$$\begin{array}{rccc} \varphi_{(A,\Phi)} : & \mathrm{Cong}\,(A, \Phi) & \rightarrow & \mathfrak{I}(A, \Phi) \\ & R & \mapsto & \{g \in A | (g, 0) \in R\} \end{array}$$

We claim that $\varphi_{(A,\Phi)} : \mathrm{Cong}(A, \Phi) \rightarrow \mathfrak{I}(A, \Phi)$ is an isomorphism of lattices.

Given $\{R_i | i \in I\} \subseteq \mathrm{Cong}\,(A, \Phi)$, it is easy to see that:

$$\psi_{(A,\Phi)}\left(\bigwedge\{R_i | i \in I\}\right) = \bigwedge \psi_{(A,\Phi)}(R_i),$$

so $\psi_{(A,\Phi)}$ is a homomorphism of lattices. Also, given $R, S \in \mathrm{Cong}\,(A, \Phi)$ such that $R \subseteq S$, we have:

$$\psi_{(A,\Phi)}(R) \subseteq \psi_{(A,\Phi)}(S).$$

Given $I' \supseteq \psi_{(A,\Phi)}(R)$, since ψ'_A is surjective, there is some $S \in \mathrm{Cong}\,(A,\Phi)$ with $\psi'_A(S) = I'$. Also, since ψ'_A is injective, such an S is unique.

Now,

$$\psi_{(A,\Phi)}(R) \subseteq \psi_{(A,\Phi)}(S),$$

so

$$R = \psi^{\dashv}_{(A,\Phi)}[\psi_{(A,\Phi)}(R)] \subseteq \psi_{(A,\Phi)}{}^{\dashv}[\psi_{(A,\Phi)}(S)] = S$$

Since $\psi_{(A,\Phi)}$ is bijective, it follows that $\psi_{(A,\Phi)}$ is an isomorphism of complete lattices. Moreover, both lattices are **algebraic lattices**, whose compact elements are the finitely generated congruences or ideals.

The following result relates the ideals of a product of $\mathcal{C}^\infty-$rings with the ideals of its factors.

Proposition 4.17. *Let A and B be two $\mathcal{C}^\infty-$rings and let $\mathfrak{I}(A)$ be the set of all ideals of A and $\mathfrak{I}(B)$ be the set of all ideals of B. Every ideal of the product $A \times B$ has the form $\mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a}$ is an ideal of A and $\mathfrak{b}$ is an ideal of B, so we have the following bijection:*

$$\begin{array}{rccc} \Phi : & \mathfrak{I}(A) \times \mathfrak{I}(B) & \rightarrow & \mathfrak{I}(A \times B) \\ & (\mathfrak{a}, \mathfrak{b}) & \mapsto & \mathfrak{a} \times \mathfrak{b} \end{array}$$

Proof. See **Proposition 18**, p. 69 of [9]. □

In the following proposition we are going to describe how to calculate any limit and any directed colimit, making use of the forgetful functor $U : \mathcal{C}^\infty$**Ring** $\rightarrow$ **Set**.

Proposition 4.18. *In the category $\mathcal{C}^\infty$**Ring** of the $\mathcal{C}^\infty-$rings, all the limits and all filtered colimits exist and are created by the forgetful functor $U : \mathcal{C}^\infty$**Ring** $\rightarrow$ **Set**.*

Proof. See p. 7 of [1]. □

By a general argument, it can be shown that the category $\mathcal{C}^\infty$**Ring** has all small colimits. In particular, coequalizers of pairs of $\mathcal{C}^\infty-$homomorphisms, $f, g : (A,\Phi) \rightarrow (B,\Psi)$, are given by quotients:

$$(A,\Phi) \underset{g}{\overset{f}{\rightrightarrows}} (B,\Psi) \xrightarrow{q_I} \left(\frac{B}{I}, \overline{\Psi}\right)$$

where $I = \langle \{(f(a), g(a)) | a \in A\} \rangle$.

In order to describe all small colimits, it is enough to construct coproducts, and since $\mathcal{C}^\infty$**Ring** has filtered colimits, it suffices to construct only finite coproducts. Also, since $\mathbb{R} \cong \mathcal{C}^\infty(\{*\})$ is the initial $\mathcal{C}^\infty-$ring, it is enough, by induction, to describe binary coproducts in $\mathcal{C}^\infty$**Ring**.

4.2. The $\mathcal{C}^\infty-$Coproduct

In this subsection we describe the coproduct in the category $\mathcal{C}^\infty$**Ring**, which E. Dubuc calls "the $\mathcal{C}^\infty-$tensor product", and we call "the $\mathcal{C}^\infty-$coproduct". First we give its categorial definition, then we define the binary $\mathcal{C}^\infty-$coproduct of free, finitely generated and finally the binary $\mathcal{C}^\infty-$coproduct of arbitrary $\mathcal{C}^\infty-$rings. Then we give a description of the $\mathcal{C}^\infty-$coproduct of an arbitrary family of arbitrary $\mathcal{C}^\infty-$rings.

Definition 4.19. Let (A, Φ) and (B, Ψ) be two $\mathcal{C}^\infty-$rings. We will denote the underlying set of the **coproduct** of (A, Φ) and (B, Ψ) by $A \otimes_\infty B$, and its corresponding canonical arrows by ι_A and ι_B:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] & \\
& A \otimes_\infty B \\
B \arrow[ur, "\iota_B"'] &
\end{tikzcd}$$

In order to describe concretely the coproduct in $\mathcal{C}^\infty$**Ring**, first we compute the coproduct of two free $\mathcal{C}^\infty-$rings with m and n generators.

Since $m = \{0, \cdots, m-1\}$, $n = \{0, \cdots, n-1\}$, $m \sqcup n \cong m + n$ and the functor $L :$ **Set** $\to \mathcal{C}^\infty$**Ring** preserves coproducts (since it is a left adjoint functor), we have:

$$\mathcal{C}^\infty(\mathbb{R}^m) \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^n) \cong \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n) \cong \mathcal{C}^\infty(\mathbb{R}^{m+n})$$

Now, given ideals $I \subseteq \mathcal{C}^\infty(\mathbb{R}^m)$ and $J \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$, then:

$$\frac{\mathcal{C}^\infty(\mathbb{R}^m)}{I} \otimes_\infty \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{J} \cong \frac{\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n)}{(I, J)},$$

where $(I, J) = \langle f \circ \pi_1, g \circ \pi_2 | (f \in I) \& (g \in J) \rangle$, with $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are the projections on the first and the second coordinates.

The empty coproduct of $\mathcal{C}^\infty-$rings is given as:

$$\bigotimes^\infty_{i \in \varnothing} A_i = \mathbb{R}$$

and the 1-ary coproduct is given by:

$$\bigotimes_{\infty} A = A$$

Now we describe concretely the finite coproduct of $\mathcal{C}^{\infty}-$rings.

We begin by defining binary coproducts. Given any two $\mathcal{C}^{\infty}-$rings, (A, Φ) and (B, Ψ), we proceed as follows:

First we write (A, Φ) and (B, Ψ) as colimits of their finitely generated $\mathcal{C}^{\infty}-$subring (according to **Theorem 3.5**):

$$(A, \Phi) \cong \varinjlim_{i \in I} (A_i, \Phi_i)$$

and

$$(B, \Psi) \cong \varinjlim_{j \in I} (B_j, \Psi_j)$$

Then, observing that colimits commute with coproducts, we have:

$$A \otimes_{\infty} B \cong \varinjlim_{\substack{i \in I \\ j \in J}} A_i \otimes_{\infty} B_j$$

Given any $n \in \mathbb{N}$ and given n $\mathcal{C}^{\infty}-$rings, $A_1, \cdots, A_n$, we define:

$$\bigotimes_{\substack{\infty \\ i=1}}^{n} A_i = A_1 \otimes_{\infty} \left(\bigotimes_{\substack{\infty \\ i=1}}^{n-1} A_i \right)$$

Now let $\{(A_i, \Phi_i) | i \in I\}$ be any set of $\mathcal{C}^{\infty}-$rings. As mentioned in **Remark 2.22** of **Subsection 2.4** , such a family has a coproduct in $\mathcal{C}^{\infty}$**Ring**. This coproduct is given by the colimit:

$$\bigotimes_{\substack{\infty \\ i \in I}} A_i = \varinjlim_{I' \subseteq_{\mathrm{fin}} I} \bigotimes_{\substack{\infty \\ i \in I'}} A_i$$

4.3. Addition of Variables: The $\mathcal{C}^{\infty}-$Ring of Polynomials

As an application of the construction given above, we can describe the process of "adding a set S of variables to a $\mathcal{C}^{\infty}-$ring (A, Φ)". The construction is given as follows:

Let (A, Φ) be any $\mathcal{C}^{\infty}-$ring and let S be any set. Consider $L(S) = \mathcal{C}^{\infty}(\mathbb{R}^S)$, the free $\mathcal{C}^\infty-$ring on the set S of generators, together with its canonical map, $\jmath_S : S \to \mathcal{C}^\infty(\mathbb{R}^S)$. If we denote by:

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] & \\
& A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S) \\
\mathcal{C}^\infty(\mathbb{R}^S) \arrow[ur, "\iota_{\mathcal{C}^\infty(\mathbb{R}^S)}"'] &
\end{tikzcd}$$

the coproduct of A and $\mathcal{C}^\infty(\mathbb{R}^S)$, define:

$$x_s := \iota_{\mathcal{C}^\infty(\mathbb{R}^S)}(\jmath_S(s)).$$

We thus define:

$$A\{x_s | s \in S\} := A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S).$$

We have a natural bijection:

$$\frac{\mathcal{C}^\infty(\mathbb{R}^S) \to A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S)}{S \to U(A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S))}$$

Thus, for each $\mathcal{C}^\infty-$ring B, each $\mathcal{C}^\infty-$homomorphism $h : A \to B$ and each function $f : S \to U(B)$ (which induces a unique $\mathcal{C}^\infty-$homomorphism $\hat{f} : \mathcal{C}^\infty(\mathbb{R}^S) \to B$) there is a unique $\mathcal{C}^\infty-$homomorphism $g : A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S) \to B$ such that $g \circ \iota_A = h$ and $\hat{f} = g \circ \iota_{\mathcal{C}^\infty(\mathbb{R}^S)}$

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"] \arrow[drr, bend left, "h"] & & \\
& A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S) \arrow[r, dashed, "\exists! g"] & B \\
\mathcal{C}^\infty(\mathbb{R}^S) \arrow[ur, "\iota_{\mathcal{C}^\infty(\mathbb{R}^S)}"'] \arrow[urr, bend right, "\hat{f}"'] & &
\end{tikzcd}$$

The $\mathcal{C}^\infty-$ring of polynomial has important applications, as to provide a guarantee of the existence of many constructions using $\mathcal{C}^\infty-$rings, such as the $\mathcal{C}^\infty-$ring fractions - which motivates us to a more detailed investigation, explored in [10].

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