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New convex integral inequalities involving multiple functions*

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Abstract. *This article explores a modern counterpart to the classical Jensen integral inequality, which provides an upper bound for convex functions evaluated at an integral. We extend this result to more general settings involving sums and products of integrals of multiple functions. Full details of the proofs are provided, and some examples illustrate the theory.*

Keywords – *Convexity; Integral inequalities; Jensen integral inequality.*
MSC2020 – *26D15*

1. Introduction

The concept of convexity lies at the heart of optimization theory, the study of inequalities, and the analysis of functional properties. To introduce its role in this article, we formally define the concept below for a class of non-negative functions defined on the half-line $[0, +\infty)$. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be convex if and only if, for any $x, y \geq 0$ and $\lambda \in [0, 1]$, the following inequality holds:

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

See [1, 2, 3]. Note that this definition can be extended to any intervals, say $[a, b]$ with $a < b$ such that $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$. For the purposes of this article, however, we will focus only on $[0, +\infty)$. If φ is twice differentiable, a classical and convenient characterization of its convexity is that its second derivative satisfies

$$\varphi''(t) \geq 0$$

for any $t \geq 0$. This condition guarantees that the first derivative φ' is non-decreasing, which further illustrates the nature of convexity. If we also assume that $\varphi(0) = 0$, then the convexity of φ implies

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that, for any $x \geq 0$ and $\lambda \in [0,1]$, we have $\varphi(\lambda x) \leq \lambda\varphi(x)$. See again [1, 2, 3]. The concept of convexity holds profound significance across many areas of mathematics and its applications. See, for example, [4, 5, 6, 7, 8, 9, 10]. In particular, the interplay between the convexity of a function and how it behaves when combined with integrals of other functions is a central theme in analysis. The best-known result in this area is the classical Jensen integral inequality, a simplified version of which is stated below. Let $f : [0,1] \to [0,+\infty)$ be a function. Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a convex function. Then the following inequality holds:

$$\varphi\left(\int_0^1 f(t)dt\right) \leq \int_0^1 \varphi\left(f(t)\right) dt,$$

provided that the last integral converges.

This article focuses on a modern counterpart of the Jensen inequality, which was first introduced in [11, 12]. It thus provides an upper bound for a convex function evaluated at the integral of a bounded function. This upper bound is expressed in terms of an integral involving the derivative of the convex function. Consequently, it serves as a complementary result to the classical Jensen integral inequality, providing an additional tool for estimating such expressions. The precise statement is given formally below. Let $f : [0,1] \to [0,+\infty)$ be a function such that, for any $t \in [0,1]$, $f(t) \leq 1$. Let $\varphi : [0,1] \to [0,+\infty)$ be a twice differentiable and convex function such that $\varphi(0) = 0$. Then the following inequality holds:

$$\varphi\left(\int_0^1 f(t)dt\right) \leq \int_0^1 f(t)\varphi'(t)dt, \tag{1}$$

provided that the last integral converges.

In this article, we present two generalizations of this result, each of which extends the scope to multiple functions. The first involves sums of integrals of functions and focuses on a quantity of the following form:

$$\varphi\left(\sum_{i=1}^{n}\int_0^1 f_i(t)dt\right),$$

where $n \in \mathbb{N}\backslash\{0\}$ and $f_1,\ldots,f_n$ are n functions that satisfy certain conditions regarding regularity and boundedness, which will be defined precisely later. We establish an upper bound in line with the approach in [11, 12]. The case $n = 1$ reduces to the inequality in Equation (1), whereas the other case yields to new results. The second generalization involves products of integrals of functions and focuses on a quantity of the following form:

$$\varphi\left(\prod_{i=1}^{n}\int_0^1 f_i(t)dt\right).$$

We provide an upper bound in line with that in [11, 12]. The case $n = 1$ reduces to the inequality in Equation (1), whereas the other case yields to new results. Full details of the proofs are provided, along with some illustrative examples for each generalization. These results improve our understanding of how convexity operates within the context of integral composition, as well as expanding the range of tools available for analyzing such expressions.

The rest of the article is as follows: Section 2 presents our two generalizations in the form of two theorems. Section 3 provides a conclusion.

2. Two theorems

2.1. First theorem

Our first theorem is presented below, followed by the detailed proof.

Theorem 2.1. *Let $n \in \mathbb{N} \backslash \{0\}$, $f_1, \ldots, f_n : [0,1] \to [0,+\infty)$ be n functions such that, for any $i = 1, \ldots, n$ and $t \in [0,1]$, we have*

$$f_i(t) \leq 1.$$

Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a twice differentiable and convex function such that $\varphi(0) = 0$. Then the following inequality holds:

$$\varphi\left(\sum_{i=1}^{n} \int_0^1 f_i(t)dt\right) \leq \frac{1}{n} \int_0^n \left(\sum_{i=1}^{n} f_i\left(\frac{t}{n}\right)\right) \varphi'(t)dt,$$

provided that the last integral converges.

Proof of Theorem 2.1. Using the differentiation rules for compositions and products of several functions, and noting that, for any $i = 1, \ldots, n$ and $t \in [0,1]$, $0 \leq f_i(t) \leq 1$, and that $\varphi'(t)$ is non-decreasing because φ is twice differentiable and convex, for any $x \in [0,1]$, we obtain

$$\begin{aligned}
&\left(\varphi\left(\sum_{i=1}^{n} \int_0^x f_i(t)dt\right)\right)' = \left(\sum_{i=1}^{n} \int_0^x f_i(t)dt\right)' \varphi'\left(\sum_{i=1}^{n} \int_0^x f_i(t)dt\right) \\
&= \left(\sum_{i=1}^{n} f_i(x)\right) \varphi'\left(\sum_{i=1}^{n} \int_0^x f_i(t)dt\right) \\
&\leq \left(\sum_{i=1}^{n} f_i(x)\right) \varphi'\left(\sum_{i=1}^{n} \int_0^x dt\right) \\
&= \left(\sum_{i=1}^{n} f_i(x)\right) \varphi'(nx).
\end{aligned}$$

Using the integration rules for compositions, $\varphi(0) = 0$, the previous result, and the change of variables $u = nx$, we get

$$\begin{aligned}
\varphi\left(\sum_{i=1}^{n}\int_{0}^{1} f_i(t)dt\right) &= \int_{0}^{1}\left(\varphi\left(\sum_{i=1}^{n}\int_{0}^{x} f_i(t)dt\right)\right)' dx + \varphi\left(\sum_{i=1}^{n}\int_{0}^{0} f_i(t)dt\right)\\
&= \int_{0}^{1}\left(\varphi\left(\sum_{i=1}^{n}\int_{0}^{x} f_i(t)dt\right)\right)' dx + \varphi(0)\\
&= \int_{0}^{1}\left(\varphi\left(\sum_{i=1}^{n}\int_{0}^{x} f_i(t)dt\right)\right)' dx\\
&\leq \int_{0}^{1}\left(\sum_{i=1}^{n} f_i(x)\right)\varphi'(nx)\,dx\\
&= \frac{1}{n}\int_{0}^{n}\left(\sum_{i=1}^{n} f_i\left(\frac{u}{n}\right)\right)\varphi'(u)du.
\end{aligned}$$

Standardizing the notation, this can also be written as follows:

$$\varphi\left(\sum_{i=1}^{n}\int_{0}^{1} f_i(t)dt\right) \leq \frac{1}{n}\int_{0}^{n}\left(\sum_{i=1}^{n} f_i\left(\frac{t}{n}\right)\right)\varphi'(t)dt.$$

This concludes the proof of Theorem 2.1. □

We highlight some particular examples of this theorem below.

- If we set $n = 1$, then we have

$$\varphi\left(\int_{0}^{1} f_1(t)dt\right) \leq \int_{0}^{1} f_1(t)\varphi'(t)dt.$$

  This is the original inequality in [11, 12].
- If we set $n = 2$, then we have

$$\varphi\left(\int_{0}^{1} f_1(t)dt + \int_{0}^{1} f_2(t)dt\right) \leq \frac{1}{2}\int_{0}^{2}\left(f_1\left(\frac{t}{2}\right) + f_2\left(\frac{t}{2}\right)\right)\varphi'(t)dt.$$

  To illustrate this inequality, we can consider $\varphi(t) = t^{\alpha}$, $t \geq 0$, with $\alpha > 1$, which satisfies the required assumption. We therefore have

$$\left(\int_{0}^{1} f_1(t)dt + \int_{0}^{1} f_2(t)dt\right)^{\alpha} \leq \frac{\alpha}{2}\int_{0}^{2}\left(f_1\left(\frac{t}{2}\right) + f_2\left(\frac{t}{2}\right)\right)t^{\alpha-1}dt.$$

- If we set $n = 3$, then we have

$$\begin{aligned}
&\varphi\left(\int_{0}^{1} f_1(t)dt + \int_{0}^{1} f_2(t)dt + \int_{0}^{1} f_3(t)dt\right)\\
&\leq \frac{1}{3}\int_{0}^{3}\left(f_1\left(\frac{t}{3}\right) + f_2\left(\frac{t}{3}\right) + f_3\left(\frac{t}{3}\right)\right)\varphi'(t)dt.
\end{aligned}$$

As an example, considering $\varphi(t) = t^\alpha$, $t \geq 0$, with $\alpha > 1$, we have

$$\left(\int_0^1 f_1(t)dt + \int_0^1 f_2(t)dt + \int_0^1 f_3(t)dt\right)^\alpha \leq \frac{\alpha}{3}\int_0^3 \left(f_1\left(\frac{t}{3}\right) + f_2\left(\frac{t}{3}\right) + f_3\left(\frac{t}{3}\right)\right) t^{\alpha-1}dt.$$

2.2. Second theorem

Our second theorem is presented below, followed by the detailed proof.

Theorem 2.2. *Let $n \in \mathbb{N}\backslash\{0\}$, $f_1, \ldots, f_n : [0,1] \to [0,+\infty)$ be n functions such that, for any $i = 1, \ldots, n$ and $t \in [0,1]$, we have*

$$f_i(t) \leq 1.$$

Let $\varphi : [0,1] \to [0,+\infty)$ be a twice differentiable and convex function such that $\varphi(0) = 0$. Then the following inequality holds:

$$\varphi\left(\prod_{i=1}^n \int_0^1 f_i(t)dt\right) \leq \frac{1}{n}\int_0^1 \left(\sum_{i=1}^n f_i(t^{1/n})\right)\varphi'(t)dt,$$

provided that the last integral converges.

Proof of Theorem 2.2. Using the differentiation rules for compositions and products of several functions, and noting that, for any $i = 1, \ldots, n$ and $t \in [0,1]$, $0 \leq f_i(t) \leq 1$, and that $\varphi'(t)$ is non-decreasing because φ is twice differentiable and convex, for any $x \in [0,1]$, we obtain

$$\begin{aligned}
&\left(\varphi\left(\prod_{i=1}^n \int_0^x f_i(t)dt\right)\right)' \\
&= \left(\prod_{i=1}^n \int_0^x f_i(t)dt\right)' \varphi'\left(\prod_{i=1}^n \int_0^x f_i(t)dt\right) \\
&= \left(\sum_{j=1}^n f_j(x)\left(\prod_{\substack{i=1 \\ i\neq j}}^n \int_0^x f_i(t)dt\right)\right)\varphi'\left(\prod_{i=1}^n \int_0^x f_i(t)dt\right) \\
&\leq \left(\sum_{j=1}^n f_j(x)\left(\prod_{\substack{i=1 \\ i\neq j}}^n \int_0^x dt\right)\right)\varphi'\left(\prod_{i=1}^n \int_0^x dt\right) \\
&= \left(\sum_{j=1}^n f_j(x)x^{n-1}\right)\varphi'\left(x^n\right).
\end{aligned}$$

Using the integration rules for compositions, $\varphi(0) = 0$, the previous result, and the change of variables $u = x^n$, we get

$$\varphi\left(\prod_{i=1}^{n}\int_0^1 f_i(t)dt\right) = \int_0^1 \left(\varphi\left(\prod_{i=1}^{n}\int_0^x f_i(t)dt\right)\right)' dx + \varphi\left(\prod_{i=1}^{n}\int_0^0 f_i(t)dt\right)$$
$$= \int_0^1 \left(\varphi\left(\prod_{i=1}^{n}\int_0^x f_i(t)dt\right)\right)' dx + \varphi(0)$$
$$= \int_0^1 \left(\varphi\left(\prod_{i=1}^{n}\int_0^x f_i(t)dt\right)\right)' dx$$
$$\leq \int_0^1 \left(\sum_{j=1}^{n} f_j(x)x^{n-1}\right) \varphi'\left(x^n\right) dx$$
$$= \frac{1}{n}\int_0^1 \left(\sum_{j=1}^{n} f_j(u^{1/n})\right) \varphi'(u)du.$$

Standardizing the notation, this can also be written as follows:

$$\varphi\left(\prod_{i=1}^{n}\int_0^1 f_i(t)dt\right) \leq \frac{1}{n}\int_0^1 \left(\sum_{i=1}^{n} f_i(t^{1/n})\right) \varphi'(t)dt.$$

This ends the proof of Theorem 2.2. □

We highlight some particular examples of this theorem below.

- If we set $n = 1$, then we have

$$\varphi\left(\int_0^1 f_1(t)dt\right) \leq \int_0^1 f_1(t)\varphi'(t)dt.$$

  This is the original inequality in [11, 12].
- If we set $n = 2$, then we have

$$\varphi\left(\int_0^1 f_1(t)dt \int_0^1 f_2(t)dt\right) \leq \frac{1}{2}\int_0^1 \left(f_1(\sqrt{t}) + f_2(\sqrt{t})\right) \varphi'(t)dt.$$

  As an example, considering $\varphi(t) = t^\alpha$, $t \in [0, 1]$, with $\alpha > 1$, we have

$$\left(\int_0^1 f_1(t)dt \int_0^1 f_2(t)dt\right)^\alpha \leq \frac{\alpha}{2}\int_0^1 \left(f_1(\sqrt{t}) + f_2(\sqrt{t})\right) t^{\alpha-1}dt.$$

- If we set $n = 3$, then we have

$$\varphi\left(\int_0^1 f_1(t)dt \int_0^1 f_2(t)dt \int_0^1 f_3(t)dt\right)$$
$$\leq \frac{1}{3}\int_0^1 \left(f_1(t^{1/3}) + f_2(t^{1/3}) + f_3(t^{1/3})\right) \varphi'(t)dt.$$

As an example, considering $\varphi(t) = t^\alpha$, $t \in [0,1]$, with $\alpha > 1$, we have

$$\left(\int_0^1 f_1(t)dt \int_0^1 f_2(t)dt \int_0^1 f_3(t)dt\right)^\alpha \leq \frac{\alpha}{3}\int_0^1 \left(f_1(t^{1/3}) + f_2(t^{1/3}) + f_3(t^{1/3})\right) t^{\alpha-1}dt.$$

3. Conclusion

Based on the results presented in [11, 12], we have improved our understanding of the relationship between convexity and integral operations by proposing new generalizations. More precisely, our generalizations extend classical inequalities to settings involving sums and products of integrals of multiple functions, thereby opening up avenues for further analysis and application research. Future work could involve investigating tighter bounds or extending these ideas to broader classes of functions, measures or domains. Such developments could have implications for fields such as probability theory, optimization, and functional inequalities in mathematical physics.

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