



Inequalities associated with certain operators within the framework of weighted Fock spaces

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Received: 13 January 2026 • Accepted: 18 June 2026 • Published: 2 July 2026

Abstract

In this paper we introduce a weighted Fock space \mathcal{F}_β . This space which gives a generalization of some Hilbert spaces of analytic functions on the complex plane \mathbb{C} like, the classical Fock space \mathcal{F} , the Dunkl type Fock space \mathcal{F}_ν and the Bessel–Struve type Fock space \mathbb{F}_ν , it plays a background to our contribution. Especially, we examine the extremal functions for the difference operator $Df(z) := \frac{1}{z^2}(f(z) - zf'(0) - f(0))$, and we deduce approximate inversion formulas for the operator D on the weighted Fock space \mathcal{F}_β .

Keywords: Analytic functions; weighted Fock spaces; extremal functions.

MSC2020: 30H20, 32A15

1 Introduction

Tikhonov regularization, named for Andrey Tikhonov, is a method of regularization of ill-posed problems. In statistics, the method is also known as ridge regression, it is particularly useful to mitigate the problem of multicollinearity in linear regression, which commonly occurs in models with large numbers of parameters [7]. In general, this method related to the Levenberg–Marquardt algorithm for solving nonlinear least squares problems. Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations from the work of Andrey Tikhonov [19, 20, 21] and David L. Phillips [10]. Some authors use the term Tikhonov–Phillips regularization.

Before the applications to the Tikhonov regularization, we shall examine the concept of the Moore–Penrose generalized inverses from the viewpoint of the theory of reproducing kernels. Here, we will be able to realize the natural and powerful method of the theory of reproducing kernels for the best approximation problems that lead to the Moore–Penrose generalized inverses.

Let E be an arbitrary set and let H_K be a reproducing kernel Hilbert space admitting the reproducing kernel K on E . For any Hilbert space H we consider a bounded linear operator T from H_K to H . Then the following problem is a classical and fundamental problem which is known as best approximate mean square norm problems

$$\inf_{f \in H_K} \left\{ \|Tf - h\|_H^2 \right\}, \tag{1.1}$$



where $h \in H$ is given. If there exists $F_T^*(h) \in H_K$ which attains this infimum, the problem (1.1) is called solvable otherwise it is called unsolvable. If H_K is a reproducing kernel Hilbert space admitting a reproducing kernel $K(p, q)$ on a the set E then whether the problem (1.1) is solvable or not, the following problem

$$\inf_{f \in H_K} \left\{ \lambda \|f\|_{H_K}^2 + \|Tf - h\|_H^2 \right\}, \quad (1.2)$$

is always solvable for all $\lambda > 0$ and we obtain a method for determine the extremal function $F_{\lambda, T}^*(h) \in H_K$ which attains the infimum (1.2).

The problem (1.2) is called the Tikhonov regularization for the problem (1.1) and if the problem (1.1) is solvable then we have

$$F_{\lambda, T}^*(h) \longrightarrow F_T^*(h) \quad \text{as } \lambda \longrightarrow 0^+,$$

in H_K and $F_T^*(h)$ is the element which attains the infimum (1.1).

Let H be a Hilbert space and $T : \mathcal{F}_\beta \rightarrow H$ be a bounded linear operator. The main goal of the paper is to find the minimizers for the following extremal problem:

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \lambda \|f\|_{\mathcal{F}_\beta}^2 + \|Tf - h\|_H^2 \right\},$$

where $h \in H$ and $\lambda > 0$. Here \mathcal{F}_β are a weighted Fock spaces of analytic functions. These spaces and some important operators on them were studied in detail by Shields [15] and later by many followers and, for instance, we can and in [4, 11] a standard references for them. We expect that the results of this paper will be useful when discussing the extremal function associated to the bounded linear operator T . As applications, we come up with some results regarding the Dunkl type Fock space \mathcal{F}_ν and the Bessel–Struve type Fock space \mathbb{F}_β .

Fock space \mathcal{F} (see [1]) is the Hilbert space of analytic functions f on \mathbb{C}^d such that

$$\|f\|_{\mathcal{F}}^2 := \frac{1}{\pi^d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dx dy < \infty, \quad z = x + iy.$$

The space \mathcal{F} is called also Segal–Bargmann space [2] and it was the aim of many works [3, 23]. Recently the author of the paper studied the extremal functions for some operators on the Fock space \mathcal{F} (see [17]).

In this paper, we introduce the weighted Fock space \mathcal{F}_β , which is the set of all analytic functions f in \mathbb{C} , with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|_{\mathcal{F}_\beta}^2 := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty,$$

where $\beta = \{\beta_n\}$ is a positive sequence so that

$$\limsup_{n \rightarrow \infty} (\beta_n)^{-1/n} = 0.$$

The space \mathcal{F}_β is a reproducing kernel Hilbert space (RKHS) that gives a generalization of some Hilbert spaces of analytic functions in the complex plane \mathbb{C} like, the Fock space \mathcal{F} (see [1, 3, 23]), the Dunkl type Fock space \mathcal{F}_ν (see [16]), and the Bessel–Struve type Fock space \mathbb{F}_ν (see [18]).

Let $D : \mathcal{F}_\beta \rightarrow \mathcal{F}_\beta$ be the difference operator given by

$$Df(z) := \frac{1}{z^2} (f(z) - zf'(0) - f(0)), \quad f \in \mathcal{F}_\beta.$$

The main goal of the paper is to find the minimizer (denoted by $F_{\lambda,D}^*(h)$) for the extremal problem:

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \lambda \|f\|_{\mathcal{F}_\beta}^2 + \|Df - h\|_{\mathcal{F}_\beta}^2 \right\},$$

where $h \in \mathcal{F}_\beta$ and $\lambda > 0$. We prove that the extremal function $F_{\lambda,D}^*(h)$ is given by

$$F_{\lambda,D}^*(h)(z) = \langle h, \Psi_z \rangle_{\mathcal{F}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\bar{z})^{n+2} w^n}{\lambda \beta_{n+2} + \beta_n}, \quad w \in \mathbb{C}.$$

Moreover, we establish approximate inversion formulas for the difference operator D on the weighted Fock space \mathcal{F}_β . A pointwise approximate inversion formulas for the operator D are also discussed.

Recently, the analog results are also proved, for the Fock space \mathcal{F} (see [17]), for the Dunkl type Fock space \mathcal{F}_ν (see [16]), and for the Bessel–Struve type Fock space \mathbb{F}_β (see [18]).

The paper is organized as follows. In Section 2 we introduce the weighted Fock space \mathcal{F}_β . In Section 3 we examine the extremal functions for the difference operator D . In Section 4, we establish approximate inversion formulas for the operator D on the weighted Fock space \mathcal{F}_β . In the last section, we summarize the obtained results and describe the open questions.

2 Weighted Fock space

We consider a sequence $\beta = \{\beta_n\}$, with $\beta_n > 0$, such that

$$\limsup_{n \rightarrow \infty} (\beta_n)^{-1/n} = 0.$$

The weighted Fock space \mathcal{F}_β is the set of all analytic functions f in \mathbb{C} , with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|_{\mathcal{F}_\beta}^2 := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{\mathcal{F}_\beta} = \sum_{n=0}^{\infty} \beta_n a_n \bar{b}_n,$$

where $f, g \in \mathcal{F}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. The set $\left\{ \frac{z^n}{\sqrt{\beta_n}} \right\}_{n=0}^{\infty}$ forms a Hilbert’s basis for the space \mathcal{F}_β . The function $K_{\mathcal{F}_\beta, z}$, $z \in \mathbb{C}$, given by

$$K_{\mathcal{F}_\beta, z}(w) := \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{\beta_n}, \quad w \in \mathbb{C},$$

is a reproducing kernel for the weighted Fock space \mathcal{F}_β .

If $\beta_n = n!$, the corresponding weighted Fock space is the Segal–Bargmann space \mathcal{F} (see [1]). This Hilbert space has the reproducing kernel

$$K_{\mathcal{F}, z}(w) = e^{w\bar{z}}, \quad w, z \in \mathbb{C}.$$

If $\beta_n = \frac{2^n ([n/2]!) \Gamma([\frac{n+1}{2}] + \nu + 1)}, $\nu \geq -1/2$, where $[n/2]$ is the integer part of $n/2$, the corresponding weighted Fock space is the Dunkl type Fock space \mathcal{F}_ν (see [16]). This Hilbert space has the reproducing kernel$

$$K_{\mathcal{F}_\nu, z}(w) = E_\nu(w\bar{z}), \quad w, z \in \mathbb{C},$$

where E_ν is the Dunkl kernel given by

$$E_\nu(w) := j_\nu(iw) + \frac{w}{2(\nu + 1)} j_{\nu+1}(iw).$$

Here

$$j_\nu(z) := 2^\nu \Gamma(\nu + 1) \frac{J_\nu(z)}{z^\nu},$$

where J_ν is the Bessel function [5, 22].

If $\beta_n = \frac{\sqrt{\pi} n! \Gamma(\frac{n}{2} + \nu + 1)}{\Gamma(\nu + 1) \Gamma(\frac{n+1}{2})}$, $\nu \geq -1/2$, the corresponding weighted Fock space is the Bessel–Struve type Fock space \mathbb{F}_ν (see [18]). This Hilbert space has the reproducing kernel

$$K_{\mathbb{F}_\nu, z}(w) = S_\nu(w\bar{z}), \quad w, z \in \mathbb{C},$$

where S_ν is the Bessel–Struve kernel given by

$$S_\nu(w) := j_\nu(iw) - i h_\nu(iw), \quad w \in \mathbb{C}.$$

Here

$$h_\nu(z) := 2^\nu \Gamma(\nu + 1) \frac{\mathbf{H}_\nu(z)}{z^\nu},$$

where \mathbf{H}_ν is the Struve function [5, 22].

In the next of this paper, we suppose that there exists a constant $c > 0$ such the sequence $\{\beta_n\}$ satisfies the condition

$$\frac{\beta_n}{\beta_{n-2}} \geq c, \quad \forall n \geq 2. \quad (2.1)$$

The condition (2.1) is verified in the precedent three cases: in the classical Fock space \mathcal{F} , in the Dunkl type Fock space \mathcal{F}_ν and in the Bessel–Struve type Fock space \mathbb{F}_ν .

3 Difference operator

Tikhonov regularization in statistics is the method of ridge regression. In general, this method related to the Levenberg–Marquardt algorithm for solving nonlinear least squares problems. Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations [20, 21].

Let \mathcal{H} be a Hilbert space, and let $T : \mathcal{F}_\beta \rightarrow \mathcal{H}$ be a bounded linear operator from \mathcal{F}_β into \mathcal{H} . Let $\lambda > 0$. We denote by $\langle \cdot, \cdot \rangle_{\lambda, \mathcal{F}_\beta}$ the inner product defined on the space \mathcal{F}_β by

$$\langle f, g \rangle_{\lambda, \mathcal{F}_\beta} := \lambda \langle f, g \rangle_{\mathcal{F}_\beta} + \langle Tf, Tg \rangle_{\mathcal{H}}.$$

The two norms $\|\cdot\|_{\mathcal{F}_\beta}$ and $\|\cdot\|_{\lambda, \mathcal{F}_\beta}$ are equivalent. In particular, we have

$$|f(z)| \leq \|f\|_{\lambda, \mathcal{F}_\beta} \left[\frac{K_{\mathcal{F}_\beta, z}(z)}{\lambda} \right]^{1/2}, \quad f \in \mathcal{F}_\beta, z \in \mathbb{C}.$$

Then the space \mathcal{F}_β , equipped with the norm $\|\cdot\|_{\lambda, \mathcal{F}_\beta}$ has a reproducing kernel $K_{\lambda, \mathcal{F}_\beta, z}$. Therefore, we have the functional equation

$$(\lambda I + T^*T)K_{\lambda, \mathcal{F}_\beta, z} = K_{\mathcal{F}_\beta, z}, \quad z \in \mathbb{C}, \tag{3.1}$$

where I is the unit operator and $T^* : \mathcal{H} \rightarrow \mathcal{F}_\beta$ is the adjoint of T .

For any $h \in \mathcal{H}$ and for any $\lambda > 0$, we define the extremal function $F_{\lambda, T}^*(h)$ by

$$F_{\lambda, T}^*(h)(z) = \langle h, TK_{\lambda, \mathcal{F}_\beta, z} \rangle_{\mathcal{H}}, \quad z \in \mathbb{C}.$$

Then by (3.1) we deduce that

$$\begin{aligned} F_{\lambda, T}^*(h)(z) &= \langle T^*h, K_{\lambda, \mathcal{F}_\beta, z} \rangle_{\mathcal{F}_\beta} \\ &= \langle T^*h, (\lambda I + T^*T)^{-1}K_{\mathcal{F}_\beta, z} \rangle_{\mathcal{F}_\beta} \\ &= \langle (\lambda I + T^*T)^{-1}T^*h, K_{\mathcal{F}_\beta, z} \rangle_{\mathcal{F}_\beta}. \end{aligned}$$

Hence

$$F_{\lambda, T}^*(h)(z) = (\lambda I + T^*T)^{-1}T^*h(z), \quad z \in \mathbb{C}. \tag{3.2}$$

The extremal function $F_{\lambda, T}^*(h)$ is the unique solution (see [12], Theorem 2.5, Section 2) of the Tikhonov regularization problem

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \lambda \|f\|_{\mathcal{F}_\beta}^2 + \|Tf - h\|_{\mathcal{H}}^2 \right\}.$$

Let D be the difference operator defined for $f \in \mathcal{F}_\beta$ by

$$Df(z) := \frac{1}{z^2}(f(z) - zf'(0) - f(0)).$$

The difference operator D is also studied in [6, 17, 18]. For $f \in \mathcal{F}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$Df(z) := \sum_{n=0}^{\infty} a_{n+2} z^n. \tag{3.3}$$

We emphasize the inversion is meaningful only on $\{f : f(0) = f'(0) = 0\}$. From (2.1), the operator D maps continuously from \mathcal{F}_β into \mathcal{F}_β , and

$$\|Df\|_{\mathcal{F}_\beta} \leq \frac{1}{\sqrt{c}} \|f\|_{\mathcal{F}_\beta}.$$

Building on the ideas of Saitoh [12, 13, 14] we examine the extremal function associated with the difference operator D .

Theorem 3.1. (i) For $f \in \mathcal{F}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$D^*f(z) = \sum_{n=2}^{\infty} \frac{\beta_{n-2}}{\beta_n} a_{n-2} z^n, \quad D^*Df(z) = \sum_{n=2}^{\infty} \frac{\beta_{n-2}}{\beta_n} a_n z^n.$$

(ii) For any $h \in \mathcal{F}_\beta$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \lambda \|f\|_{\mathcal{F}_\beta}^2 + \|Df - h\|_{\mathcal{F}_\beta}^2 \right\}$$

has a unique extremal function given by

$$F_{\lambda,D}^*(h)(z) = \langle h, \Psi_z \rangle_{\mathcal{F}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\bar{z})^{n+2} w^n}{\lambda \beta_{n+2} + \beta_n}, \quad w \in \mathbb{C}.$$

Proof. (i) If $f, g \in \mathcal{F}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle Df, g \rangle_{\mathcal{F}_\beta} = \sum_{n=0}^{\infty} \beta_n a_{n+2} \bar{b}_n = \sum_{n=2}^{\infty} \beta_{n-2} a_n \bar{b}_{n-2} = \langle f, D^*g \rangle_{\mathcal{F}_\beta},$$

where

$$D^*g(z) = \sum_{n=2}^{\infty} \frac{\beta_{n-2}}{\beta_n} b_{n-2} z^n.$$

And therefore

$$D^*Df(z) = \sum_{n=2}^{\infty} \frac{\beta_{n-2}}{\beta_n} a_n z^n.$$

(ii) We put $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $F_{\lambda,D}^*(h)(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3.2) we have $(\lambda I + D^*D)F_{\lambda,D}^*(h)(z) = D^*h(z)$. By (i) we deduce that

$$c_0 = c_1 = 0, \quad c_n = \frac{\beta_{n-2} h_{n-2}}{\lambda \beta_n + \beta_{n-2}}, \quad n \geq 2.$$

Thus

$$F_{\lambda,D}^*(h)(z) = \sum_{n=0}^{\infty} \frac{\beta_n h_n}{\lambda \beta_{n+2} + \beta_n} z^{n+2} = \langle h, \Psi_z \rangle_{\mathcal{F}_\beta}, \quad (3.4)$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\bar{z})^{n+2} w^n}{\lambda \beta_{n+2} + \beta_n}, \quad w \in \mathbb{C}.$$

The theorem is proved. □

If \mathcal{F}_β is the classical Fock space \mathcal{F} . For $f \in \mathcal{F}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$D^*f(z) = \sum_{n=2}^{\infty} \frac{a_{n-2}}{n(n-1)} z^n, \quad D^*Df(z) = \sum_{n=2}^{\infty} \frac{a_n}{n(n-1)} z^n.$$

And for any $h \in \mathcal{F}$ and for any $\lambda > 0$, one has

$$F_{\lambda,D}^*(h)(z) = \langle h, \Psi_z \rangle_{\mathcal{F}},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\bar{z})^{n+2} w^n}{n! [\lambda(n+2)(n+1) + 1]}.$$

If \mathcal{F}_β is the Dunkl type Fock space \mathcal{F}_ν . For $f \in \mathcal{F}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$D^*f(z) = \sum_{n=1}^{\infty} \frac{a_{2n-2}}{4n(n+\nu)} z^{2n} + \sum_{n=1}^{\infty} \frac{a_{2n-1}}{4n(n+\nu+1)} z^{2n+1},$$

$$D^*Df(z) = \sum_{n=1}^{\infty} \frac{a_{2n}}{4n(n+\nu)} z^{2n} + \sum_{n=1}^{\infty} \frac{a_{2n+1}}{4n(n+\nu+1)} z^{2n+1}.$$

And for any $h \in \mathcal{F}_\nu$, and for any $\lambda > 0$, one has

$$F_{\lambda,D}^*(h)(z) = \langle h, \Psi_z \rangle_{\mathcal{F}_\nu}, \quad \Psi_z(w) = \Psi_z^e(w) + \Psi_z^o(w),$$

where

$$\Psi_z^e(w) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)(\bar{z})^{2n+2} w^{2n}}{2^{2n} n! [4\lambda(n+1)(n+\nu+1) + 1] \Gamma(n+\nu+1)},$$

$$\Psi_z^o(w) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)(\bar{z})^{2n+3} w^{2n+1}}{2^{2n+1} n! [4\lambda(n+1)(n+\nu+2) + 1] \Gamma(n+\nu+2)}.$$

If \mathcal{F}_β is the Bessel–Struve type Fock space \mathbb{F}_ν . For $f \in \mathbb{F}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$D^*f(z) = \sum_{n=2}^{\infty} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+\nu)}{n(n-1)\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2}+\nu+1)} a_{n-2} z^n,$$

$$D^*Df(z) = \sum_{n=2}^{\infty} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+\nu)}{n(n-1)\Gamma(\frac{n-1}{2})\Gamma(\frac{n}{2}+\nu+1)} a_n z^n.$$

And for any $h \in \mathbb{F}_\nu$, and for any $\lambda > 0$, one has

$$F_{\lambda,D}^*(h)(z) = \langle h, \Psi_z \rangle_{\mathcal{B}_\nu},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)\Gamma(\frac{n+1}{2})(\bar{z})^{n+2} w^n}{\sqrt{\pi} n! [\lambda(n+2)(n+2\nu+2) + 1] \Gamma(\frac{n}{2}+\nu+1)}.$$

4 Approximate inversion formulas

In this section we establish the estimate properties of the extremal function $F_{\lambda,D}^*(h)(z)$, and we deduce approximate inversion formulas for the difference operator D . These formulas are the analogous of Calderón’s reproducing formulas for the Fourier type transforms [8, 9]. A pointwise approximate inversion formulas for the operator D are also discussed.

The extremal function $F_{\lambda,D}^*(h)$ given by (3.4) satisfies the following properties.

Lemma 4.1. *If $\lambda > 0$ and $h \in \mathcal{F}_\beta$, then*

- (i) $|F_{\lambda,D}^*(h)(z)| \leq \frac{1}{2\sqrt{\lambda}} (K_{\mathcal{F}_\beta,z}(z))^{1/2} \|h\|_{\mathcal{F}_\beta},$
- (ii) $|DF_{\lambda,D}^*(h)(z)| \leq \frac{1}{2\sqrt{\lambda c}} (K_{\mathcal{F}_\beta,z}(z))^{1/2} \|h\|_{\mathcal{F}_\beta},$
- (iii) $\|F_{\lambda,D}^*(h)\|_{\mathcal{F}_\beta} \leq \frac{1}{2\sqrt{\lambda}} \|h\|_{\mathcal{F}_\beta}.$

Proof. Let $\lambda > 0$ and $h \in \mathcal{F}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (3.4) we have

$$|F_{\lambda,D}^*(h)(z)| \leq \|\Psi_z\|_{\mathcal{F}_\beta} \|h\|_{\mathcal{F}_\beta}.$$

Using the fact that $(x + y)^2 \geq 4xy$ we obtain

$$\|\Psi_z\|_{\mathcal{F}_\beta}^2 = \sum_{n=0}^{\infty} \beta_n \left[\frac{|z|^{n+2}}{\lambda\beta_{n+2} + \beta_n} \right]^2 \leq \frac{1}{4\lambda} \sum_{n=0}^{\infty} \frac{|z|^{2(n+2)}}{\beta_{n+2}} \leq \frac{1}{4\lambda} K_{\mathcal{F}_\beta, z}(z).$$

This gives (i).

On the other hand, from (3.3) and (3.4) we have

$$DF_{\lambda, D}^*(h)(z) = \sum_{n=0}^{\infty} \frac{\beta_n h_n}{\lambda\beta_{n+2} + \beta_n} z^n = \langle h, \Phi_z \rangle_{\mathcal{F}_\beta}, \quad (4.1)$$

where

$$\Phi_z(w) = \sum_{n=0}^{\infty} \frac{(w\bar{z})^n}{\lambda\beta_{n+2} + \beta_n}.$$

Then

$$|DF_{\lambda, D}^*(h)(z)| \leq \|\Phi_z\|_{\mathcal{F}_\beta} \|h\|_{\mathcal{F}_\beta}.$$

And by (2.1) we deduce that

$$\|\Phi_z\|_{\mathcal{F}_\beta}^2 = \sum_{n=0}^{\infty} \beta_n \left[\frac{|z|^n}{\lambda\beta_{n+2} + \beta_n} \right]^2 \leq \frac{1}{4\lambda} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta_{n+2}} \leq \frac{1}{4\lambda c} K_{\mathcal{F}_\beta, z}(z).$$

This gives (ii).

Finally, from (3.4) we have

$$\|F_{\lambda, D}^*(h)\|_{\mathcal{F}_\beta}^2 = \sum_{n=2}^{\infty} \beta_n \left[\frac{\beta_{n-2}|h_{n-2}|}{\lambda\beta_n + \beta_{n-2}} \right]^2.$$

Then we obtain

$$\|F_{\lambda, D}^*(h)\|_{\mathcal{F}_\beta}^2 \leq \frac{1}{4\lambda} \sum_{n=2}^{\infty} \beta_{n-2} |h_{n-2}|^2 = \frac{1}{4\lambda} \|h\|_{\mathcal{F}_\beta}^2,$$

which gives (iii) and completes the proof of the lemma. \square

We establish approximate inversion formulas for the difference operator D .

Theorem 4.2. *If $\lambda > 0$ and $h \in \mathcal{F}_\beta$, then*

- (i) $\lim_{\lambda \rightarrow 0^+} \|DF_{\lambda, D}^*(h) - h\|_{\mathcal{F}_\beta} = 0,$
- (ii) $\lim_{\lambda \rightarrow 0^+} \|F_{\lambda, D}^*(Dh) - h_0\|_{\mathcal{F}_\beta} = 0,$ where $h_0(z) = h(z) - zh'(0) - h(0).$

Proof. Let $\lambda > 0$ and $h \in \mathcal{F}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (4.1) we have

$$DF_{\lambda, D}^*(h)(z) - h(z) = \sum_{n=0}^{\infty} \frac{-\lambda\beta_{n+2}h_n}{\lambda\beta_{n+2} + \beta_n} z^n.$$

Therefore

$$\|DF_{\lambda, D}^*(h) - h\|_{\mathcal{F}_\beta}^2 = \sum_{n=0}^{\infty} \beta_n \left[\frac{\lambda\beta_{n+2}|h_n|}{\lambda\beta_{n+2} + \beta_n} \right]^2$$

Again, by dominated convergence theorem and the fact that

$$\beta_n \left[\frac{\lambda\beta_{n+2}|h_n|}{\lambda\beta_{n+2} + \beta_n} \right]^2 \leq \beta_n |h_n|^2,$$

we deduce (i).

Finally, from (3.3) and (3.4) we have

$$F_{\lambda,D}^*(Dh)(z) - h_0(z) = \sum_{n=2}^{\infty} \frac{-\lambda\beta_n h_n}{\lambda\beta_n + \beta_{n-2}} z^n.$$

So, one has

$$\|F_{\lambda,D}^*(Dh) - h_0\|_{\mathcal{F}_\beta}^2 = \sum_{n=2}^{\infty} \beta_n \left[\frac{\lambda\beta_n |h_n|}{\lambda\beta_n + \beta_{n-2}} \right]^2.$$

Using the dominated convergence theorem and the fact that

$$\beta_n \left[\frac{\lambda\beta_n |h_n|}{\lambda\beta_n + \beta_{n-2}} \right]^2 \leq \beta_n |h_n|^2,$$

we deduce (ii). □

Since \mathcal{F}_β is a reproducing kernel Hilbert space, the evaluation function is always bounded on \mathcal{F}_β . Then, from Theorem 4.2, we deduce the following pointwise approximate inversion formulas for the operator D .

Corollary 4.3. *If $\lambda > 0$ and $h \in \mathcal{F}_\beta$, then*

- (i) $\lim_{\lambda \rightarrow 0^+} DF_{\lambda,D}^*(h)(z) = h(z),$
- (ii) $\lim_{\lambda \rightarrow 0^+} F_{\lambda,D}^*(Dh)(z) = h_0(z).$

5 Conclusion and perspective

We investigated the extremal function $F_{\lambda,D}^*(h)$ related to the weighted Fock space \mathcal{F}_β , which is the solution of the Tikhonov regularization problem

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \lambda \|f\|_{\mathcal{F}_\beta}^2 + \|Df - h\|_{\mathcal{F}_\beta}^2 \right\}. \tag{5.1}$$

Some results related to this function on \mathbb{C} are proven, such as

$$F_{\lambda,D}^*(h)(z) = (\lambda I + D^*D)^{-1} D^*h(z), \quad z \in \mathbb{C}. \tag{5.2}$$

The function $F_{\lambda,D}^*$ also obeys to the following conclusions.

▷ Let $\delta, \lambda > 0$ and $h, h_\delta \in \mathcal{F}_\beta$ such that $\|h - h_\delta\|_{\mathcal{F}_\beta} \leq \delta$. Then by Lemma 4.1 (iii), we have

$$\|F_{\lambda,D}^*(h) - F_{\lambda,D}^*(h_\delta)\|_{\mathcal{F}_\beta} \leq \frac{\delta}{2\sqrt{\lambda}}.$$

When h, h_δ contain errors or noises, we need error estimates for (5.2).

▷ Let $h \in \mathcal{F}_\beta$, in the limit case $\lambda \rightarrow 0$, the problem (5.1) reduces to the Tikhonov regularization problem

$$\inf_{f \in \mathcal{F}_\beta} \left\{ \|Df - h\|_{\mathcal{F}_\beta}^2 \right\},$$

and its extremal function is defined as

$$F_{0,D}^*(h)(z) = \lim_{\lambda \rightarrow 0} F_{\lambda,D}^*(h)(z), \quad z \in \mathbb{C}.$$

And from Corollary 4.3 (i), for $h \in \mathcal{F}_\beta$, we obtain

$$DF_{0,D}^*(h)(z) = \lim_{\lambda \rightarrow 0} DF_{\lambda,D}^*(h)(z) = h(z).$$

On the other hand, from Corollary 4.3 (ii), for $h \in \mathcal{F}_\beta$, we obtain

$$F_{0,D}^*(Dh)(z) = \lim_{\lambda \rightarrow 0^+} F_{\lambda,D}^*(Dh)(z) = h_0(z) = h(z) - zh'(0) - h(0).$$

We emphasize the pointwise inversion formula is meaningful only on $\{h : h(0) = h'(0) = 0\}$.

In the future work, by using computers, we shall illustrate numerical experiments approximation formulas for the difference operator D in the case of some examples of weighted Fock spaces \mathcal{F}_β , when $\lambda \rightarrow 0$.

Conflict of interest. The author declares that there is no conflict of interest regarding the publication of this paper.

Acknowledgments. The author wishes to thank the Reviewers for their careful reading and editing of the paper.

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