Model Theory Inspired by Grothendieckian Algebraic Geometry: a Survey of Sheaf Representations for Categorical Model Theory*

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Abstract. *In this survey, we expound sheaf representations of categories in the context of categorical logic. Namely, we present classifying topoi of coherent theories in terms of equivariant sheaves over a topological groupoid, show a generalization of this technique using localic groupoids and finally expose a representation of Grothendieck topoi as global sections of sheaf. Finally, we apply these techniques to provide a quick glance at logical schemes, a novel theory proposed as the model-theoretic analogue to the schemes of Algebraic Geometry.*

Keywords – *Sheaves; Grothendieck Topoi; Categorical Logic; Schemes.*
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Introduction

This survey is intended to readers with basic skills in both Category Theory and in Logic, especially those that are interested in their interface: the vernacular Categorical Logic, a field but half a century old. The final aim of this paper is to introduce the reader to Breiner's Logical Schemes ([4]), interesting objects proposed around ten years ago to be to Model Theory as Schemes are to Algebraic Geometry.

Naturally, given the fundamental nature of sheaf theory in the development of schemes, we shall also need to study sheaf representations in the context of categorical logic. To be specific, we shall need to traverse the seminal works of Joyal and Tierney ([19]), Moerdijk ([21]), Awodey [2], and more, which have been established as relevant

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to Topos Theory. Therefore, even if the reader is not terribly interested in Categorical Logic they might find this survey (specifically sections 2.2 and 2.3) as a convenient tool in studying (certain aspects of the multifaceted notion of) topoi.

One might wonder if Breiner's core proposal, namely, the search of a Scheme for Model Theory, is a reasonable one. Why, this desire is permitted when one realizes the proximity of the core objects studied: algebraic varieties may be seen as definable subset indeed. A well know voicing of this sentiment was made by Wilfrid Hodges, who opens his compendium [17] with the equation "Model Theory = Algebraic Geometry - Fields". Still this epigram concerns "Classical"(i.e., pre-Grothendieck, pre-Scheme, pre-functor even) Algebraic Geometry; this begs: what about Grothendieckian Algebraic Geometry? Under Lawvere's "Functors are models" doctrine this question should have been tantalizing, nonetheless "the formal methods of the two fields are radically different" as Breiner remarks. The rise of categorical logic in the last century, especially the developments regarding topoi, give us hope perhaps of an approximation of techniques of both fields by the inclusion of Grothendieck's sheaf theoretic methods.

Finally, let us quickly sketch the paper: we start, for the convenience of the reader, with a quick section of preliminaries on pretopoi and "theories as categories". The bulk of this project, section 2, is then spent developing the tools needed to attempt Breiner's aim. Namely, we devote it to the representation of categories by sheaves over groupoids. Keeping in mind the slogan "theories are categories with structures", we are thus talking of representing theories as sheaves. To be more specific, we start section 2 by expounding the work of Henrik Forssell ([13], [1], [14], [15]), who obtains a explicit representation of coherent theories by topological groupoids. Next, we expand the aforementioned representation to a more general categorical context: following the work of Carsten Butz and Ieke Moerdijk ([6], [7], [8]), we study a topological groupoid representation of Grothendieck topos with sufficient points. Following, we make a quick detour to the theory of descent so we have the machinery to achieve our most general representation: the celebrated result of Joyal and Tierney presenting Grothendieck Topoi as classifying topoi of localic groupoids. To finish with this rather technical section, we further reap or foray into descent theory to obtain Awodey's sheaf representation of topoi, cf. [2], which improves the results of Moerdijk and Butz. We then move to the final section, concluding the work with our stated objective of expounding Spencer Breiner's thesis, [4], presenting the notion of logical schemes and showing some promising results.

1. Preliminaries

In this first section we recall, for the reader's convenience, some preliminaries for the work. Namely, we include some basic facts of pretopoi and categorical model theory.

We assume that the reader is familiar with the basic language and concepts of category theory and logic. For completeness, we have added an appendix on categorical syntax and semantics at the end of the paper.

1.1. Pretopoi

Definition 1.1. We say that a category $\mathcal{C}$ *has images* if, and only if, for object y the inclusion functor $Sub(y) \hookrightarrow \mathcal{C}/y$ admits a left adjoint $im(-)$. A category is said to be *coherent* whenever it $i)$ has finite limits; $ii)$ has images; $iii)$ has pullback-stable regular epimorphisms (i.e., epimorphisms that happen as coequalizers) and $iv)$ its subobject lattices have pullback-stable unions. A functor between coherent categories is called *coherent* iff it preserves finite limits, regular epimorphisms and unions. We denote by $\mathbf{Coh}$ the category of coherent functors between coherent categories.

Definition 1.2. We will need the following definitions,

- Given objects $A, B \in \mathcal{C}$ we say the coproduct $A \coprod B$ to be *disjoint* iff the inclusions $A \rightarrowtail A \coprod B$ and $B \rightarrowtail A \coprod B$ are monic and its meet (in $Sub(A \sqcup B)$) is the initial object. We say *positive* a coherent category in which all of the coproducts are disjoint.
- Given subobject $R \rightarrowtail A^2$ in a category $\mathcal{C}$ with finite limits, we say it *an equivalence relation* iff, for each $U \in \mathcal{C}$, the induced subset $Hom(U, R) \subseteq Hom(U, A^2)$ is an equivalence relation. It is routine to check that any kernel pair is an equivalence. We say a category in which all equivalence relations happen as kernel pairs to be *effective*.

A *pretopos* is an effective positve coherent category. We denote by $\mathbf{PTopos}$ the full subcategory of $\mathbf{Coh}$ whose objects are pretopoi.

Theorem 1.3 (Pretopos Completion)**.** *The inclusion functor* $\mathbf{PTopos} \hookrightarrow \mathbf{Coh}$ *admits left adjoint* $PTop(-)$*. Furthermore, the unit* $\mathcal{C} \to PTop(\mathcal{C})$ *is conservative, full and full on subobjects*[1].

Proof: See [11, A, 1.4.5, 3.3.10] □

Remark 1.4. As the inclusion is conservative, we shall identify a coherent $\mathcal{C}$ with its image in $PTop(\mathcal{C})$.

Let us now speak of factorizations for pretopoi morphisms.

Definition 1.5. We say a functor between pretopos $F : \mathcal{P} \to \mathcal{Q}$ to be a *quotient* iff $i)$ for every $B \in \mathcal{Q}$ there is $A \in \mathcal{P}$ with epimorphism $IA \twoheadrightarrow B$ and $ii)$ F is full on subobjects.

[1] That is, if for any $B \leq IA$ there is $A' \in \mathcal{P}$ with $IA' \cong B$.

Proposition 1.6 (Conservative-Quotient Factorization)**.** *Every functor between pretopos $F : \mathcal{P} \to \mathcal{Q}$ admits a factorization as quotient followed by a conservative functor. Furthermore, quotient morphisms are orthogonal to conservative morphisms.*

Proof: See [4, 2.2.4]. □

Remark 1.7. We note that the coherent sheaf functor, $\mathrm{Sh}_{Coh}(-)$, sends the conservative-quotient factorization of coherent morphisms into the well-know surjective-embedding factorization of geometric morphisms.

1.2. Theories and Categories

We now show a way to obtain a category from a theory and vice versa. We use the standard notation of the field but, if needed, the reader may consult the appendix.

Definition 1.8. Given contexts $\overline{x} = (x_1, x_2, \cdots, x_n)$ and $\overline{y} = (y_1, y_2, \cdots, y_n)$, we say the formulas $\overline{x}.\varphi$ and $\overline{y}.\psi$ to be *α-equivalent* if we may obtain ψ by replacing free occurrences of x_i by y_i in φ. Given a coherent theory $\mathbb{T}$, its *syntactical category*, denototed by $\mathbf{Syn}(\mathbb{T})$, has as objects the equivalence classes of α-equivalent coherent formulas $[\overline{x}.\varphi]$ and as morphisms[2] $[\overline{x}.\varphi] \to [\overline{y}.\psi]$ equivalence classes $[\overline{x}, \overline{y}.\theta]$ of $\mathbb{T}$-provably functional formulas between φ and ψ, that is to say, such that the sequences below are provable[3] in $\mathbb{T}$

$$\varphi \vdash_{\overline{x}} \exists \overline{y}(\theta) \quad \theta \vdash_{\overline{x},\overline{y}} \varphi \wedge \psi \quad \theta \wedge \theta' \vdash_{\overline{x},\overline{y},\overline{z}} \overline{y} = \overline{z}.$$

With θ' as θ where every free instance of y_i has been replaced by z_i, for $\overline{z}$ any context disjoint from $\overline{x}$ and $\overline{y}$.

Proposition 1.9. $\mathbf{Syn}(\mathbb{T})$ *is a coherent category.*

Proof: See [12, D, 1.4.10]. □

Proposition 1.10. *Given a coherent theory $\mathbb{T}$, we have an equivalence natural in $\mathcal{D}$*

$$\mathbf{Coh}(\mathbf{Syn}(\mathbb{T}), \mathcal{D}) \simeq \mathbb{T}\text{-}\boldsymbol{Mod}(\mathcal{D})$$

for $\mathcal{D}$ a coherent category. Analogously, setting $\mathcal{P}[\mathbb{T}]$ as the pretopos completion (cf. 1.3) of $\mathbf{Syn}(\mathbb{T})$, we have an equivalence natural in $\mathcal{Q}$

$$\mathbf{PTopos}(\mathcal{P}[\mathbb{T}], \mathcal{Q}) \simeq \mathbb{T}\text{-}\boldsymbol{Mod}(\mathcal{Q})$$

[2]We assume $\overline{x}$ and $\overline{y}$ disjoint. As we are considering the formulas modulo α-equivalence there is no loss of generality.

[3]We say a sequent σ to be provable in $\mathbb{T}$ iff for every $M \in \mathbb{T}Mod(\mathbf{Set})$ we have $M \vDash \sigma$.

for $\mathcal{Q}$ a pretopos. In special, we have an equivalence $\mathbf{PTopos}(\mathcal{P}[\mathbb{T}], \mathcal{Q}) \simeq \mathbf{Coh}(\mathcal{P}[\mathbb{T}], \mathcal{Q})$*, natural in the pretopos $\mathcal{Q}$.*

Proof (sketch): See [*loc. cit.*, D, 1.4.12] for the details. We mention that the half $\mathbb{T}$-$\mathbf{Mod}(\mathcal{Q}) \to \mathbf{PTopos}(\mathcal{P}[\mathbb{T}], \mathcal{Q})$ of the equivalence sends M to the functor whose actions is given by $[\varphi] \mapsto [\![\varphi]\!]^M$. The remaining half, by Yoneda, corresponds to evaluating the functor $\mathcal{P}[\mathbb{T}] \to \mathcal{Q}$ in some generic object $U \in \mathcal{P}[\mathbb{T}]$, which we baptize as the *generic model of* $\mathbb{T}$. Analogous considerations apply to the coherent case. □

We now take the inverse path of the one in definition 1.8, we show how to define a theory from a category.

Definition 1.11. Given a small coherent category $\mathcal{C}$, define a signature $\Sigma_{\mathcal{C}}$ by adjoining

- For each $c \in \mathcal{C}_0$, one type $\lceil c \rceil$;
- For each subobject $R : \prod_n A_i \rightarrowtail B$, a relation $\lceil R \rceil : \prod_n \lceil A_i \rceil \leq \lceil B \rceil$.
- For each morphism $f : \prod_n A_i \to B$, a function $\lceil f \rceil : \prod_n \lceil A_i \rceil \to \lceil B \rceil$.

Define now the theory $\mathbb{T}_{\mathcal{C}}$ over $\Sigma_{\mathcal{C}}$, whose sequents are

- $\top \vdash_x \lceil 1_c \rceil(x) = x$, for each $c \in \mathcal{C}_0$;
- $\top \vdash_x \lceil gf \rceil(x) = \lceil g \rceil(\lceil f \rceil(x))$, for each $c \xrightarrow{f} d \xrightarrow{g} e$ $in \mathcal{C}$;
- $\top \vdash_{[]} \exists x(\top)$ and $\top \vdash_{x,y} x = y$, for each pair of variables x and y of type $\lceil 1 \rceil$, with 1 the terminal object of $\mathcal{C}$;
- $\top \vdash_x [\lceil h \rceil(\lceil f \rceil(x)) = \lceil k \rceil(\lceil f \rceil(g))]$, $[\lceil h \rceil(x) = \lceil k \rceil(y)] \vdash_{x,y} [\exists z(\lceil f \rceil(z) = y) \wedge (\lceil g \rceil(z) = y)]$ and $[(\lceil f \rceil(x) = \lceil f \rceil(y)) \wedge (\lceil g \rceil(x) = \lceil g \rceil(y))] \vdash_{x,y} x = y$, braces added for readability, for each pullback in $\mathcal{C}$ as below
$$\begin{array}{ccc} A \times_C B & \xrightarrow{k} & A \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$
- $\top \vdash_x \bigvee_n (\exists y_i(\lceil f_i \rceil(y_i) = x))$, for each jointly surjective finite family $\{f_i : U_i \to U\}_n$

Proposition 1.12. *Given coherent categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ that preserves finite limits is coherent iff it preserves jointly epimorphic finite families*

Proof: We recall that a functor is coherent iff it is regular and preserves finite unions. Also, that a functor is regular iff it preserves finite limits and regular epimorphisms. Note that in a regular category, regular epimorphisms coincide with epimorphisms left-orthogonal to monomorphisms. Since F preserves jointly epic finite families, in particular, it preserves epimorphisms. Furthermore, by preserving finite limits, the functor preserves monomorphisms. It is clear then that if an epimorphism is left-orthogonal to a monomorphism then so too will be its image by F, therefore, F is regular. Finally, note that a finite family of subobjects $\{U_i \rightarrowtail U\}_{i<n}$ is jointly epic iff $\bigvee_{i<n} U_i = U$, and so, F preserves unions. □

Proposition 1.13. *For any coherent category $\mathcal{C}$, we have $\mathbf{Syn}(\mathbb{T}_\mathcal{C}) \simeq \mathcal{C}$. For any pretopos $\mathcal{P}$, we have $PTop(\mathbf{Syn}(\mathbb{T}_\mathcal{P})) \simeq \mathcal{P}$.*

Proof: Given coherent category $\mathcal{D}$, the equivalence between models $\mathbb{T}_\mathcal{C}$-$\mathbf{Mod}(\mathcal{D})$ and functors $\mathcal{C} \to \mathcal{D}$ that preserve finite limits and jointly epic finite families is immediate. So, by the above lemma, we have $\mathbf{Coh}(\mathcal{C}, \mathcal{D}) \simeq \mathbb{T}_\mathcal{C}$-$\mathbf{Mod}(\mathcal{D})$. We then obtain the equivalence $\mathbf{Coh}(\mathcal{C}, \mathcal{D}) \simeq \mathbf{Coh}(\mathbf{Syn}(\mathbb{T}_\mathcal{C}), \mathcal{D})$ and, by Yoneda, we get $\mathbf{Syn}(\mathbb{T}_\mathcal{C}) \simeq \mathcal{C}$. The case for a pretopos is analogous. □

Remark 1.14. Since every coherent category happens as a syntactic category $\mathbf{Syn}(\mathbb{T})$, we may describe pretopos completion (theorem 1.3) as an operation on theories. A celebrated observation by Makkai and Reyes (cf. [20]) states the action $\mathcal{C} \mapsto PTop(\mathcal{C})$ corresponds to $\mathbf{Syn}(\mathbb{T}) \mapsto \mathbf{Syn}(\mathbb{T}^{eq})$, where $\mathbb{T}^{eq}$ is Shelah's elimination of imaginaries.

2. Sheaf Representations

In this section we expound the central ideas of the paper. We begin by showing a representation of logical theories by (equivariant sheaves of) topological groupoids. We follow Henrik Forssell's thesis, [13], and the subsequent articles, [1], [14], [15]. We then take a step back and show a categorical generalization of Forssell's representation on the work of Carsten Butz thesis, [6]. Moving on, we present a quick introduction of the theory of descent along indexed categories, which we use to obtain our final two representations: the celebrated one of Joyal and Tierney [19], and Awodey's one, [2], generalizing the results of Butz and Moerdijk.

2.1. Theories and Groupoids

Now, we explore the representation of theories in terms of (equivariant sheaves of) topological groupoids. We will see, in the following subsection, that the representation obtained here is an instance of a more general categorical construction. However, this choice is justified in view of the simplicity that the construction takes when restricted to the logical context. Indeed, while next subsection we will talk about enumeration sets, artificial objects at first sight, here we need only mention isomorphisms between the models of our theory! We follow the aforementioned thesis, [13], in addition to the articles that accompanied it, [1], [14] e [15].

To avoid size issues, we need to restrict our considerations to some choice of small models. Given the $\mathbb{T}$ theory about a Σ signature, fix a cardinal $\kappa \geq |\Sigma| \cup \omega$. Let $X_{\mathbb{T}}$ the space of $\mathbb{T}$-models whose underlying set is an element of κ, their topology being the coarsest one containing

$$\langle\langle [\overline{x}.\varphi], \overline{a}\rangle\rangle := \left\{M \in X_{\mathbb{T}} : \overline{a} \in [\![\overline{x}.\varphi]\!]^M\right\}$$

for coherent Σ-formulas φ with n free variables and lists $\overline{a} = (a_1, a_2, \cdots, a_n) \in \kappa^n$. Next, let $G_{\mathbb{T}}$ be the groupoid of isomorphisms between models. Its topology is the coarsest making the domain and codomain maps $d_0, d_1 : G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ continuous and containing, for all types A from Σ and for $a, b \in \kappa$, the sets

$$\langle\langle A, a \mapsto b\rangle\rangle := \left\{f : M \xrightarrow{\cong} N \in G_{\mathbb{T}} : a \in [\![A]\!]^M \wedge f_A(a) = b\right\}.$$

We mention in passing that the spaces $G_{\mathbb{T}}$ and $X_{\mathbb{T}}$ are sober, cf. [1, 1.2.7].

Given a formula $\overline{x}.\varphi$, consider the set $[\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}} := \left\{\langle M, \overline{a}\rangle : M \in X_{\mathbb{T}}, \overline{a} \in [\![\overline{x}.\varphi]\!]^M\right\}$ and let its topology as the coarsest making the projection $[\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}} \to X_{\mathbb{T}}$ continuous. Using the description of the basics of $X_{\mathbb{T}}$, it is clear that

Lemma 2.1. *The basics of $[\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}}$ are of the form*

$$\langle\langle [\overline{x}, \overline{y} : \psi], \overline{a}\rangle\rangle := \left\{\langle M, \overline{b}\rangle : \overline{b} \star \overline{a} \in [\![\overline{x}, \overline{y} : \varphi \wedge \psi]\!]^M\right\}$$

for coherent ψ and $\overline{a} \in \kappa^n$. Here $\overline{b} \star \overline{a}$ denotes the concatenation of $\overline{b}$ and $\overline{a}$. Hence, the map is a local homeomorphism and the action $(\overline{x}.\varphi) \mapsto [\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}}$ describes a functor $\mathbf{Syn}(\mathbb{T}) \to \mathit{Sh}(X_{\mathbb{T}})$.

We now introduce a concept that shall be central to the rest of the work.

Definition 2.2. Given a topological groupoid as below

$$\mathbf{G} = \quad G_1 \times_{G_0} G_1 \;\underset{d_{02}}{\overset{d_{01}}{\underset{- d_{12} \to}{\rightrightarrows}}}\; G_1 \;\underset{d_1}{\overset{d_0}{\underset{\leftarrow i -}{\rightrightarrows}}}\; G_0$$

An equivariant sheaf is a pair $X \in Sh(G_0)$ and $\theta : s_0^* X \to s_1^* X$ satisfying

$$i^*\theta = 1 \quad {s_{12}}^*\theta \circ {s_{01}}^*\theta \cong {s_{02}}^*\theta$$

where $s_i : \mathrm{Sh}(G_1) \to \mathrm{Sh}(G_0)$ is the geometric morphism induced by $d_i : G_1 \to G_0$ and similarly for s_{ij}. Alternatively, using the equivalence $\mathrm{Sh}(X) \simeq \text{Ét}(X)$ we can describe an equivariant sheaf as an étale spaces $E \to G_0$ associated with an unitary and associative action $\mu : G_1 \times_{G_0} E \to E$. We denote the category of equivariant sheaves for the groupoid $\mathbf{G}$ by $B\mathbf{G}$. We shall make use of both descriptions throughout this paper, with no danger of confusion.

Remark 2.3. In the literature, BG is sometimes referred to as the classifying topos of the groupoid G. A motivation for this name may be found quickly when $\mathbf{G}$ is a group (that is, when $G_0 = 1$) as a classic result of sheaf theory gives us that for any space X we have $Geom(Sh(X), BG)$ equivalent to the principal G-bundles over X, that is, BG is the classifying topos of principal G-bundles. In the general case, BG still is a classifying topos, but the result is more subtle, cf. [22, 6.1] and [5, 3.4, 5.3].

Note that we can extend lemma's 2.1 functor to $\mathcal{M} : \mathbf{Syn}(\mathbb{T}) \to BG_{\mathbb{T}}$ by setting $[\overline{x}.\varphi] \mapsto ([\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}}, \theta)$, where

$$\theta(\langle M, \overline{a}\rangle, M \xrightarrow{f} N) = \langle N, f(\overline{a})\rangle$$

We say *definable* the objects of $BG_{\mathbb{T}}$ in the image of $\mathcal{M}$.

Proposition 2.4. *The functor $\mathcal{M}$ is coherent and reflects covers of $BG_{\mathbb{T}}$ into coverages for the coherent topology. In particular, $\mathcal{M}$ is conservative and, by Diaconescu's theorem, $\mathcal{M}$ induces a geometric morphism $BG_{\mathbb{T}} \xrightarrow{m} Sh_{Coh}(\mathbf{Syn}(\mathbb{T}))$.*

Proof (sketch): Let $BG_{\mathbb{T}} \xrightarrow{U} \mathsf{Sh}(X_{\mathbb{T}}) \xrightarrow{U'} \mathbf{Set}/X_{\mathbb{T}}$ the forgetful functors. We have

$$\begin{array}{ccc} & & BG_{\mathbb{T}} \\ & \overset{\mathcal{M}}{\nearrow} & \downarrow U \\ \mathbf{Syn}(\mathbb{T}) & \longrightarrow & \mathsf{Sh}(X_{\mathbb{T}}) \\ & \searrow & \downarrow U' \\ & & \mathbf{Set}/X_{\mathbb{T}} \end{array}$$

As both the forgetful functors reflect coherent structure and coverings, it is enough to show that the action $[\overline{x}.\varphi] \mapsto ([\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}} \to X_{\mathbb{T}})$ does so also to show $\mathcal{M}$ coherent and continuous. Why, we have the triangle

$$\begin{array}{ccc} & \mathbf{Syn}(\mathbb{T}) & \\ \swarrow & & \searrow^{\prod_{X_{\mathbb{T}}} M} \\ \mathbf{Set}/X_{\mathbb{T}} & \simeq & \prod_{M \in X_{\mathbb{T}}} \mathbf{Set}/M \end{array}$$

We have then that, since $\mathbb{T}$ has enough models, $\prod_{X_{\mathbb{T}}} M$ is conservative and as all M are coherent their product is too so. Therefore, it follows $\mathbf{Syn}(\mathbb{T}) \to \mathbf{Set}/X_{\mathbb{T}}$ conservative and coherent, thus it also reflects covers, as we wanted. For more details on this argument, see [15, 3.2.2]. □

Lemma 2.5. *Given $[\overline{x}.\varphi] \in \mathbf{Syn}(\mathbb{T})$, the basic open of $[\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}}$ closed under the action θ are of the form $[\![\overline{x} : \xi]\!]_{X_{\mathbb{T}}} \subseteq [\![\overline{x}.\varphi]\!]_{X_{\mathbb{T}}}$ for some coherent formula ξ.*

Proof: Let $U = \langle\langle [\overline{x}, \overline{y} : \psi], \overline{a} \rangle\rangle$ a basic open of $[\![\overline{x} : \phi]\!]_{X_{\mathbb{T}}}$. Passing through an isomorphism if necessary (cf. [13, 2.3.4.1]), we can assume without loss of generality that for $i < j$ with y_i, y_j of the same type we have $a_i \neq a_j$. Hence, letting σ the conjunction of the inequalities $y_i \neq y_j$ for $i < j$ with y_i, y_j of the same type, I state that for $\xi = \exists \overline{y}(\sigma \wedge \psi \wedge \varphi)$ we have $[\![\overline{x} : \xi]\!]_{X_{\mathbb{T}}}$ the closure of U under the action θ. Indeed, we have $U \subseteq [\![\overline{x} : \xi]\!]_{X_{\mathbb{T}}}$ and $[\![\overline{x} : \xi]\!]_{X_{\mathbb{T}}}$ closed under θ. Furthermore, if $(M, \overline{b}) \in [\![\overline{x} : \xi]\!]_{X_{\mathbb{T}}}$ then there is $\overline{c}$ with $\overline{b} \star \overline{c} \in [\![\overline{x}, \overline{y} : \sigma \wedge \psi \wedge \varphi]\!]^M$. We can construct an isomorphism f that swaps the lists $\overline{b}$ and $\overline{c}$ in place (cf., eg, [1, 1.2.5]), and we then have $\theta((M, \overline{b}), \overline{c}) \in U$, that is, $(M, \overline{b})$ is in the closure of U under θ. □

The above lemma allows us to conclude that the subobjects of definable object are unions of definable objects. In particular, a subobject of a compact object $\mathcal{M}([\overline{x}.\varphi])$ will be of the form $\bigvee_n \mathcal{M}([\overline{x} : \psi_i])$ which, taking a disjunction, may be reduced to $\mathcal{M}([\overline{x} : \bigvee_n \psi_i])$, that is, $\mathcal{M}$ is full on compact subobjects. Therefore, since $\mathcal{M}$ reflects coverages, we can conclude that the definable objects are compact. In particular, the graph of a morphism between two definable objects will be definable and then, because $\mathcal{M}$ is coherent and conservative, it will be the graph of a morphism of $\mathbf{Syn}(\mathbb{T})$. That is, $\mathcal{M}$ is full. Furthermore, we also have that

Proposition 2.6. *Definable objects generate the topos $BG_{\mathbb{T}}$.*

Proof: See [1, 1.4.7]. □

Theorem 2.7 (Equivariant Sheaf Representation for theories)**.** *Given coherent theory $\mathbb{T}$, $BG_{\mathbb{T}}$ is its classifying topos. Specifically, the functor m of proposition 2.4 is half of an equivalence $BG_{\mathbb{T}} \simeq Sh_{Coh}(\mathbf{Syn}(\mathbb{T}))$*

Proof: By proposition 2.6, definable objects are a site of definition for $BG_{\mathbb{T}}$. Furthermore, the discussion above and the proposition 2.4 give us that $\mathcal{M}$ is fully faithful and reflects coverages, therefore, by the comparison lemma (see, eg, [12, C, 2.2.3]), we have $BG_{\mathbb{T}} \simeq \mathrm{Sh}_{Coh}(\mathbf{Syn}(\mathbb{T}))$. □

Finally, we mention in passing that we may further obtain a Galois connection between the subgroupoids and subtopoi of our representation.

Theorem 2.8 (Galois Connection)**.** *Let $\mathbb{T}$ be a coherent theory. Given a subtopos $\mathcal{H} \leq BG_{\mathbb{T}}$ there exists subtheory $\mathbb{T}' \subseteq \mathbb{T}$ with $\mathcal{H} \cong Sh(G_{\mathbb{T}'})$. Furthermore, letting $Sub(BG_{\mathbb{T}})$ the category of subtopoi of $BG_{\mathbb{T}}$ and $Sub(G_{\mathbb{T}})$ the category subgroupoids of $G_{\mathbb{T}}$, define a map $pt : Sub(BG_{\mathbb{T}}) \to Sub(G_{\mathbb{T}})$ setting*

$$pt(BG_{\mathbb{T}'}) = \{M \in (G_{\mathbb{T}})_0 : M \vDash \mathbb{T}'\}$$

Also define a map $B : Sub(G_{\mathbb{T}}) \to Sub(BG_{\mathbb{T}})$ by $B\mathcal{H} = BG_{Th(\mathcal{H})}$, where

$$Th(\mathcal{H}) = \{\varphi : \forall M \in \mathcal{H}_0(M \vDash \varphi)\}$$

The above maps form a Galois Connection in the sense that $B\mathcal{H} \leq \mathcal{F}$ if and only if $\mathcal{H} \leq pt(\mathcal{F})$.

Proof: See [14, 3.2.2]. □

2.2. Topoi and Groupoids

In this section we show a more tractable version of the representation of topoi by localic groupoids, due to Carsten Butz and Ieke Moerdijk. By restricting our attention to Grothendieck's topos with enough points, we may obtain a topological groupoid representation. We follow closely the pair of articles [7] and [8].

Let's start by explaining the notion of points and what constitutes having enough of them.

Definition 2.9. Let $\mathcal{E}$ be a Grothendieck topos. A *point* of $\mathcal{E}$ is a geometric morphism x : **Set** $\to \mathcal{E}$ and, for some object $A \in \mathcal{E}$, the *stalk* of A in x is $x^*(A)$, which we sometimes as A_x.

We say that a Grothendieck topos $\mathcal{E}$ has *enough points* if "isomorphism can be tested fiberwise", that is, if the class of points is jointly conservative,

$$\forall A \xrightarrow{f} B(\forall \textbf{Set} \xrightarrow{x} \mathcal{E}(x^* f : x^* A \cong x^* B) \implies f : A \cong B)$$

A classic result (cf. [12, C, 2.2.11]) allows us to guarantee that when $\mathcal{E}$ has enough points then $\mathcal{E}$ has a set of enough points, a fact we will make use of in what follows.

Remark 2.10. We mention that the property of enough points has an interesting logical interpretation. In short, for a topos $\mathcal{S}[\mathbb{T}]$ classifying a theory $\mathbb{T}$, as each point of $\mathcal{S}[\mathbb{T}]$ is equivalent to a model in **Set**, having enough points is equivalent to the theory having enough set models, i.e., to it satisfying a completeness theorem for set models. We notice then that the case to which we restrict our attention, of topoi with enought points, is quite common in the logical world.

With this introduction done, we can begin to define our topological space. Given a Grothendieck topos $\mathcal{E}$ with enough points let

- $\mathcal{P}_{\mathcal{E}}$ a set of jointly conservative points for $\mathcal{E}$;
- $S_{\mathcal{E}}$ an object of $\mathcal{E}$ whose subobjects of powers (i.e., the sheaves $B \leq S_{\mathcal{E}}^n$ for some natural n) generate $\mathcal{E}$, and
- $I_{\mathcal{E}}$ a cardinal such that $\text{card}(S_p) \leq I$ for every $p \in \mathcal{P}_{\mathcal{E}}$.

Remark 2.11. The object $S_{\mathcal{E}}$ always exists. We can take it as, e.g., the disjoint union of the objects in a small site of definition for $\mathcal{E}$. Alternatively, if $\mathcal{E}$ is the classifying topos of a theory then we can take $S_{\mathcal{E}}$ as the generic model for that theory.

Given a set A with $\text{card}(A) \leq I_{\mathcal{E}}$, consider the set of enumerations of A,

$$\text{En}_{I_{\mathcal{E}}}(A) = \left\{ D \xrightarrow{u} A : D \subseteq I_{\mathcal{E}}, \forall a \in A (f^{-1}(a) \text{ is infinite}) \right\}$$

We then define the space $X_{\mathcal{E}}$ as the set of enumerations of the S_p modulo isomorphisms of the enumerations, that is,

$$X_{\mathcal{E}} := \coprod_{p \in \mathcal{P}_{\mathcal{E}}} \text{En}_{I_{\mathcal{E}}}(\mathcal{S}_p) / \sim$$

Where $\sim$ is the equivalence relation generated by

$$(D_1 \xrightarrow{u} S_p, p) \sim (D_2 \xrightarrow{v} S_q, q) \iff \exists \tau (\tau : p^* \cong q^* \wedge \tau_S \circ u = v)$$

We denote the class $(u, p)/\sim$ by $[u, p]$. Next, we define a topology on $X_{\mathcal{E}}$ putting as basic open sets $U_{B,\bar{a}} := \{[u, p] \in X_{\mathcal{E}} : (u(a_1), \cdots, u(a_n)) \in B_p\}$ for each $B \leq S_{\mathcal{E}}^n$ of $\mathcal{E}$ and $\bar{a} = (a_1, a_2, \cdots, a_n) \in {I_{\mathcal{E}}}^n$.

Remark 2.12. To see how Forssell's groupoid $G_{\mathbb{T}}$ happens as a groupoid of enumeration notice that without major changes in theory we could have defined $X_{\mathbb{T}}$ as the space of κ-small models along with enumerations, as opposed to models with underlying set in κ. We mention in passing that this was the technique adopted by Spencer Breiner in his thesis, which we recommend the reader for more details on this approach.

Let us now define a morphism $\mathrm{Sh}(X_{\mathcal{E}}) \xrightarrow{\varphi} \mathcal{E}$. We begin by describing the inverse image, specifying its action on the fibers: for each E object of $\mathcal{E}$, put $(\varphi^* E)_{[u,p]} = E_p$. More explicitly we associate with each E the étale $\varphi^* E \to X_{\mathcal{E}}$, where

$$\varphi^* E = \{([u,p], e) : [u,p] \in X_{\mathcal{E}}, e \in E_p\}$$

with basic open, for $B \leq S^n$, $\bar{a} = (a_1, a_2, \cdots, a_n) \in {I_{\mathcal{E}}}^n$ e $B \xrightarrow{f} C \in \mathcal{E}$

$$V_{B,\bar{a},f} = \{([u,p], e) : [u,p] \in U_{B,\bar{a}}, e = f_p(u(\bar{a}))\}$$

and π the obvious projection. Now, observing the action on the stalks, it is clear that φ^* preserves finite limits and colimits, so we can conclude that φ^* has a right adjoint, and so, it determines a geometric morphism. We also note that the inverse image admits a left adjunct $\mathrm{Sh}(X_{\mathcal{E}}) \xrightarrow{\varphi_!} \mathcal{E}$, defined in the basics by $\varphi_!(U_{\bar{a},B}) = B$ and extended by colimits.

Lemma 2.13. *Given a point p, define a topology on $En_{I_{\mathcal{E}}}(S_p)$ by setting as basics the sets*

$$Z_{\bar{a},\bar{s}} := \{u : D \to S_p : \bigwedge_n (u(a_i) = s_i)\}$$

We claim that the diagram below commutes

$$\begin{array}{ccc} Sh(En_{I_{\mathcal{E}}}(S_p)) & \xrightarrow{i_p} & Sh(X_{\mathcal{E}}) \\ \pi \downarrow & & \downarrow \varphi \\ \textbf{\textit{Set}} & \xrightarrow[p]{} & \mathcal{E} \end{array} \quad \textit{undefined}$$

Proof: We note that the connected components of $i_p^{-1}(U_{B,\bar{a}})$ take the form of the basics $Z_{\bar{a},\bar{s}}$ for $\bar{s} \in S_p^n$ and $\bar{a}$. Now, consider the section $U_{B,\bar{a}} \to V_{B,\bar{a},f}$ given by $\sigma([u,p]) = ([u,p], f_p(u\bar{a})$ and observe that σ is constant with value $f_p(\bar{s})$ in the connected $Z_{\bar{a},\bar{s}}$. We conclude that $i_p^* \varphi^*(E)$ is the constant sheaf of fiber E_p and, as $\pi^* p^*(E) = E_p$, the result follows. □

We can now prove the following result.

Proposition 2.14. *Let $\mathcal{E}$ be a Grothendieck topos with enough points. The geometric morphism $Sh(X_{\mathcal{E}}) \xrightarrow{\varphi} \mathcal{E}$ is connected, i.e., φ^* is fully faithful.*

Proof: Let us use the notation of the square in lemma 2.13. Remember that φ^* is fully faithful iff we have an isomorphism $\varphi_* \varphi^* \cong 1$. Furthermore, by the hypothesis on $\mathcal{E}$, we need only check this isomorphism stalkwise. Given $p \in \mathcal{P}_{\mathcal{E}}$, we have

$$\begin{aligned} \varphi_! \varphi^*(X)_p &=^{\dagger} p^* \varphi_! \varphi^*(E) \\ &=^{\star} \pi_!(i_p)^* \varphi^*(X) \\ &= \pi_! \pi^*(X_p) = X_p \end{aligned}$$

Where $=^{\dagger}$ comes from the Beck-Chevalley condition (which we may use as enumeration spaces are locally connected, cf. [8, 3.3]), $=^{\star}$ from the lemma 2.13 and the last equality from the fact that every enumeration space is connected (cf. [*loc. cit.*]). □

Remark 2.15. Note that the construction of $X_{\mathcal{E}}$ depends functorially on the parameters $\mathcal{P}_{\mathcal{E}}, S_{\mathcal{E}}$ and $I_{\mathcal{E}}$. Indeed, if we expand the number of points $\mathcal{P} \subseteq \mathcal{P}'$ we have a clear inclusion functor $X_{\mathcal{E}}(\mathcal{P}) \to X_{\mathcal{E}}(\mathcal{P}')$. Analogously, an epimorphism $J \twoheadrightarrow I$ induces a functor $X_{\mathcal{E}}(I) \to X_{\mathcal{E}}(J)$ as well as a subsheaf $S \leq S'$ induces an arrow $X_{\mathcal{E}}(S') \to X_{\mathcal{E}}(S)$. In particular, given a geometric morphism $\mathcal{E} \to \mathcal{F}$ we can fix parameters $\mathcal{P}_1$ and I_1 for $\mathcal{E}$ and S_2 for $\mathcal{F}$, and then choose $\mathcal{P}_2$ and I_2 in $\mathcal{F}$ large enough so that the spaces X_1 and X_2 form the commutative diagram below, see [8, 2.4] for more details.

$$\begin{array}{ccc} \mathrm{Sh}(X_1) & \longrightarrow & \mathrm{Sh}(X_2) \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} \end{array}$$

Next, we can now describe the topological groupoid $G_{\mathcal{E}}$ that shall represent the topos $\mathcal{E}$. $G_{\mathcal{E}}$ will have, as objects, points from $X_{\mathcal{E}}$ and, as maps, isomorphisms between points from $\mathcal{E}$ respecting the enumerations. Explicitly, the points of $G_{\mathcal{E}}$ are triples "$\theta : (u,p) \to (v,q)$" for $(u,p),(v,q) \in \coprod_{\mathcal{P}_{\mathcal{E}}} \mathrm{En}(\mathcal{S}_p)$ e $\theta : p^* \cong q^*$, modulo the equivalence relation $((u,p),(v,q),\theta) \equiv ((u',p'),(v',q'),\theta')$ given by

$$\exists \alpha, \beta [(\alpha : p^* \cong (p')^* \wedge \alpha_{\mathcal{S}} \circ u = u') \wedge (\beta : q^* \cong (q')^* \wedge \beta_{\mathcal{S}} \circ v = v') \wedge (\beta\theta = \theta'\alpha)]$$

The first two conditions ensure that $[u,p] = [u',p']$ and $[v,q] = [v',q']$, while the last one that θ and θ' "preserve" these equivalences. We will denote the class $((u,p),(v,q),\theta)/\!\equiv$ by $[\theta : (u,p) \to (v,q)]$.

Remark 2.16. Note that every point of G is of the form $[id : (u,p) \to (v,p)]$. Indeed, observe that the classes $[\theta : (a,p) \to (b,q)]$ and $[id : (a,p) \to (\theta_{\mathcal{S}} \circ b, p)]$ coincide.

Next, we define a topology on $G_{\mathcal{E}}$. For each pair of subsheaves $B, C \leq S^n_{\mathcal{E}}$ and pair $\bar{a} = (a_1, \cdots, a_n), \bar{b} = (b_1, \cdots, b_n) \in I^n_{\mathcal{E}}$, define a basic open

$$W_{B,\bar{b},C,\bar{c}} = \left\{ [(u,p) \xrightarrow{\theta} (v,q)] : u(\bar{a}) \in B_p \wedge v(\bar{b}) \in C_q \wedge \theta(u(\bar{a})) = v(\bar{b}) \right\}$$

We will need the following lemma to prove our representation.

Lemma 2.17. *Let* $d_0, d_1 : BG_{\mathcal{E}} \rightrightarrows Sh(X_{\mathcal{E}})$ *be the codomain and domain maps* [4]. *The diagram below commutes and satisfies the Beck-Chevalley formula, "$d_{0!}d_1^*(F) = \varphi^*\varphi_!(F)$".*

$$\begin{array}{ccc} BG_{\mathcal{E}} & \xrightarrow{d_0} & Sh(X_{\mathcal{E}}) \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Sh(X_{\mathcal{E}}) & \xrightarrow[\varphi]{} & \mathcal{E} \end{array}$$

Proof: Given $[\theta : (u,p) \to (v,q)] \in G_{\mathcal{E}}$ and $E \in \mathcal{E}$, we have $d_0^*\varphi^*E_p = E_p$ and $d_1^*\varphi^*E_q = E_q$, and so, the fibers of the isomorphism $d_0^*\varphi^* \to d_1^*\varphi^*$ come from the isomorphism $E_p \to E_q$, induced by θ. Now, let us show that $d_{0!}d_1^*(F) = \varphi^*\varphi_!(F)$. Given $x = (u,r) \in X_{\mathcal{E}}$ consider the diagram

$$\begin{array}{ccccc} \mathrm{Sh}(\mathrm{En}(S_r)) & \xrightarrow{k_{(u,r)}} & BG_{\mathcal{E}} & \xrightarrow{d_0} & \mathrm{Sh}(X) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \varphi} \\ \mathbf{Set} & \xrightarrow[x]{} & \mathrm{Sh}(X) & \xrightarrow[\varphi]{} & \mathcal{E} \end{array}$$

For $\mathrm{En}(S_r)$ as in the lemma 2.13 and where $k_{(u,r)} : \mathrm{En}(S_r) \to G_{\mathcal{E}}$ comes from the action $v \mapsto [id : (u,r) \to (v,r)]$. Note that $k^{-1}_{(u,r)}(W_{B,\bar{b},C,\bar{c}}) = Z_{\bar{b},\overline{u(c)}}$ and so the function is continuous. Furthermore, using the fact that every point $[\theta : (u,p) \to (v,q)]$ can be written as $[id : (u,p) \to (v,p)]$, we note that $k^{-1}_{(u,r)}$ is in bijection with $d_0^{-1}((u,r))$, so we have $d_0^{-1}((u,r)) \cong \mathrm{En}(S_r)$, that is, the left square is a pullback. Next, since the groupoid $G_{\mathcal{E}}$ is locally connected (cf. [7, 4.2]), we may use the Beck-Chevalley condition to obtain $x^*(d_0)_! \cong \pi_!k^*_{(u,r)}$. Also, as $d_1.k_{(u,r)} = i_p$ and $\varphi.x = p$, the lemma 2.13 guarantees $(\varphi x)^*\varphi_! = \pi_!(d_1k_{(u,r)})^*$. Therefore, joining all the above equalities we get

$$d_{0!}{d_1}^*(F)_x = x^*d_{0!}d_1^*F = \pi_!k^*_{(u,r)}d_1^*F = \varphi^*\varphi_!(F) = \varphi^*\varphi_!(F)_x$$

As we wanted. □

We can now prove the

[4] See, because $G_{\mathcal{E}}$ is a groupoid we don't need a "respectively"!

Theorem 2.18 (Butz-Moerdijk Topoi representation)**.** *Let $\mathcal{E}$ be a Grothendieck topos with enough points. The functor $\varphi^* : \mathcal{E} \to Sh(X_{\mathcal{E}})$ described above induces an equivalence of categories $\mathcal{E} \simeq BG_{\mathcal{E}}$.*

Proof: By proposition 2.14, $\mathcal{E}$ is equivalent to the category of coalgebras for the comonad $\varphi^*\varphi_*$, that is, it is equivalent to the algebras for the monad $\varphi^*\varphi_!$ in $\mathrm{Sh}(X_{\mathcal{E}})$. Let us then show that this last category is equivalent to the equivariant sheaves over $G_{\mathcal{E}}$. Indeed, by Beck-Chevalley, $d_{0!}d_1^*(F) = \varphi^*\varphi_!(F)$ and so a morphism $\tau : \varphi^*\varphi_!(F) \to F$ is equivalent to a map $d_1^*(F) \to d_0^*(F)$ which, passing to the projection $d_0^*(F) \to F$, is equivalent to an action $\mu : d_1^*(F) \to F$. Let us show that this μ satisfies the cocycle conditions iff the initial morphism τ is an algebra. Using the lemmas 2.13 and 2.2 we have, for any $(u, p) \in X_{\mathcal{E}}$, that

$$\varphi^*\varphi_!(F)_{(u,p)} = \pi_! i_p^*(F) = \text{“set of connected components of } i_p^*(F)\text{”}$$

Hence, a point $x \in F_{(u,p)}$ defines a connected component $[x] \in i_p^*(F)$ and $\tau_{(u,p)}([x])$ defines a point of $F_{(u,p)}$. Given a point $[id : (u, p) \to (v, p)]$ of $G_{\mathcal{E}}$ and $x \in F_{(v,p)}$ we have $\mu(g, x) = \tau_{(u,p)}([x])$. If τ is an algebra, $\tau_{(u,p)}([x]) = x$ and so $\tau_{(u,p)}([\tau_{(v,p)}([x])]) = \tau_{(u,p)}([x])$, therefore, we get $\mu(1, x) = x$ and $\mu(g \circ h, x) = \mu(g, (\mu h, x))$. The reciprocal is perfectly analogous, and thus the result follows. □

Remark 2.19. We could obtain an alternative proof of the above theorem by noting that we have enough results to show φ comonadic and then using theorem 2.31 in conjunction example 2.29. While this would be “cleaner”, we wouldn’t get such an explicit description of our groupoid $G_{\mathcal{E}}$.

Remark 2.20. We can establish another relationship between the two representations we have seen. Given a coherent theory $\mathbb{T}$, Deligne’s theorem (cf. [24, IX, 11.3]) guarantees that its classifying topos, given by $\mathcal{S}[\mathbb{T}] := \mathrm{Sh}_{Coh}(\mathbf{Syn}(\mathbb{T}))$, has enough points. Thus, we may use the theorem 2.18 to obtain a groupoid $G_{\mathcal{S}[\mathbb{T}]}$ that represents the topos $\mathcal{S}[\mathbb{T}]$. The above corollary then guarantees that we will have $BG_{\mathcal{S}[\mathbb{T}]} \simeq BG_{\mathbb{T}}$, that is, that the obtained groupoids will be “Morita-equivalent”.

Remark 2.21. The main goal of this subsection was to prove the Butz-Moerdijk representation theorem above that establishes the equivalence between a Grothendieck topos and its corresponding classifying groupoid. We thank the referee for the comment, as topological groupoids are internal groupoids in the (2,1)-category of topological spaces, then many aspects of this theorem should hold, more generally, for groupoids internal to other (2,1)-categories (see, for instance, section 2 in [25]).

2.3. Descent Theory for Topoi

A quick slogan for descent theory may be as "the study of the sheaf condition (i.e. section which are coherent locally may be lifted to global section) in a higher categorical context". Just as the study sheaf condition over topological spaces (which may be regarded as 0-categorical) leads us to consider topos (which are 1-categories), the study of the sheaf condition over Grothendieck topologies leads us to the $2-$categorical realm[5]. To develop this language the situation begets we need to form a notion of "2-presheaf" – roughly a contravariant pseudofunctor into the (2-)category of groupoids instead into the category of sets ($\cong$ the category of discrete categories)– and that is the role of indexed categories. Here, Steve Awodey's thesis, [2, V], and Marta Bunge's paper, [5], are our main references.

Definition 2.22. Given category $\mathcal{E}$, an *$\mathcal{E}$-indexed category* is a pseudofunctor $\underline{A}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathfrak{CAT}$, that is to say, it consists of the following data

- For every object i of $\mathcal{E}$, a category A^i;
- For each morphism $i \xrightarrow{\alpha} j$ of $\mathcal{E}$, a functor $A^j \xrightarrow{\alpha^*} A^j$;
- For each object e of $\mathcal{E}$, a natural isomorphism $(1_e)^* \xrightarrow[\eta_e]{\cong} 1_{A^e}$;
- For each pair $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ of $\mathcal{E}$, a natural isomorphism $\alpha^*\beta^* \xrightarrow[\mu_{\alpha\beta}]{\cong} (\beta\alpha)^*$;

So that for each triple $i \xrightarrow{\alpha} j \xrightarrow{\beta} k \xrightarrow{\gamma} l$ and for $a \xrightarrow{\delta} b$, the diagrams below commute

$$\begin{array}{ccc} \alpha^*\beta^*\gamma^* & \xrightarrow{\alpha*\mu_{\beta,\gamma}} & \alpha^*(\gamma\beta)^* \\ {\scriptstyle \mu_{\alpha,\beta}\gamma^*}\downarrow & & \downarrow{\scriptstyle \mu_{\alpha,\beta\gamma}} \\ (\beta\alpha)^*\gamma^* & \xrightarrow[\mu_{\beta\alpha,\gamma}]{} & (\gamma\beta\alpha)^* \end{array} \qquad \begin{array}{ccccc} \delta^*(1_b)^* & \xrightarrow{\delta^*\eta_b} & \delta^* & \xleftarrow[\eta_a\delta^*]{} & (1_a)^*\delta^* \\ & {\scriptstyle \mu_{\delta,1_b}}\searrow & \Vert & \swarrow{\scriptstyle \mu_{1_a,\delta}} & \\ & & \delta^* & & \end{array}$$

Example 2.23. If $\mathcal{E}$ has pullbacks, we can define a canonical indexation of $\mathcal{E}$ over itself, which we shall denote by $\underline{\mathcal{E}}$, by the pullback action: we associate to $\mathcal{E}/i$ each object i and to the pullback $\alpha^* : \mathcal{E}/j \to \mathcal{E}/i$ each arrow $\alpha : i \to j$. As the pullback of the composition is isomorphic to the composition of the pullbacks, it is routine to verify that the coherence conditions are observed. Generalizing, if $F : \mathcal{E} \to \mathcal{C}$ preserves fiber products then $\mathcal{C}$ has $\mathcal{E}$-canonical indexing given by $i \mapsto \mathcal{C}/Fi$.

We will say an indexed category to be *strict* whenever the coherence isomorphisms are equalities, that is, whenever $\underline{A}_{\mathcal{E}}$ is, in fact, a functor. Conveniently, every essentially small indexed $\underline{A}_{\mathcal{E}}$ (that is, whose fibers A^i are equivalent to small categories) is equivalent to a strict indexed category.

[5] The reader can think that a 2-category is just a category enriched over the cartesian category Cat: this, in fact, defines the notion of *strict* 2-category; instead, the general notion is *bicategory*.

Lemma 2.24. *Every essentially small $\mathcal{E}$-indexed category admits a strictification.*

Proof (sketch): A full demonstration of this fact would be better placed in a work where indexed categories play a more central role, so we restrict ourselves to an outline. In short, the idea is to define an indexed category $\underline{B}_{\mathcal{E}}$ by

$$B^i := \underline{Hom}_{\mathcal{E}}([i], \underline{A}_{\mathcal{E}})$$

Where $[i]$ is the $\mathcal{E}$-indexed category given by $j \mapsto \mathcal{E}(j,i)$ and $\underline{Hom}_{\mathcal{E}}([i], A)$ denotes the category of $\mathcal{E}$-indexed natural transformations between the $\mathcal{E}$-indexed functors $[i] \to \underline{A}_{\mathcal{E}}$ or, in 2-category language, $\underline{Hom}_{\mathcal{E}}([i], A)$ is the category of modifications between the pseudonatural transformations between $[i]$ and $\underline{A}_{\mathcal{E}}$. For more details, we recommend to readers the work of [23], which has explicit descriptions of this category, along with a version of Yoneda lemma (cf. [*loc. cit.*, I.1.2]) for indexed categories which, in particular, ensures that $\underline{Hom}_{\mathcal{E}}([i], \underline{A}_{\mathcal{E}}) \simeq A^i$. Thus, $B^i \simeq A^i$ as we wanted. Finally, remembering that the action of $\underline{B}_{\mathcal{E}}$ is given by $\alpha \mapsto \mathcal{E}(-, \alpha)$, we see that $\underline{B}_{\mathcal{E}}$ is indeed strict. □

We may now speak of descent objects and stacks. Following the analogy of the introduction, if indexed categories are our 2-presheaves then stacks will be our 2-sheaves and descent objects our matching families.

Definition 2.25. Let $\mathcal{E}$ be a category with pullbacks, $\underline{A}_{\mathcal{E}}$ an $\mathcal{E}$-indexed category and $R = \{U_i \xrightarrow{r_i} U\}_I$ a family of morphisms from $\mathcal{E}$. A *descent object* for $\underline{A}_{\mathcal{E}}$ over R is a family $(c_i, \alpha_{ij})_{i,j \in I}$ where

- For every $i \in I$ we have $c_i \in A^{U_i}$;
- For every $i, j \in I$ we have $(\pi_i)^*(c_i) \xrightarrow[\alpha_{ij}]{\cong} (\pi_j)^*(c_j)$, with the maps "$\pi$" coming from the pullback below

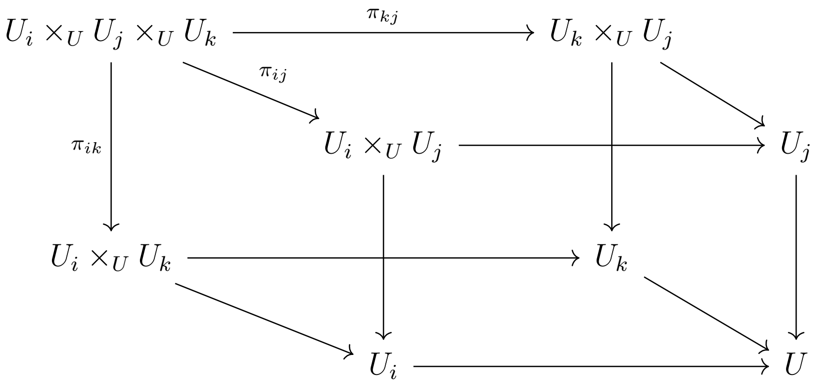
$$\begin{array}{ccc} U_i \times_U U_j & \xrightarrow{\pi_i} & U_i \\ {\scriptstyle \pi_j}\downarrow & \lrcorner & \downarrow{\scriptstyle r_i} \\ U_j & \xrightarrow[r_j]{} & U \end{array}$$

so that this data satisfies the cocycle condition:

$$\Delta_{U_i}{}^*(\alpha_{ii}) = 1_{c_i} \qquad \pi_{jk}{}^*(\alpha_{jk}).\pi_{ij}{}^*(\alpha_{ij}) = \pi_{ik}{}^*(\alpha_{ik})$$

for every $i, j, k \in I$, where Δ_{U_i} the diagonal[6] and the "π" maps come from the pullback cube

[6]The diagonal of an object X is the universal map $\Delta : X \to X^2$ with $\pi_1 \Delta_X = \pi_0 \Delta_X = 1_X$



We can now form **Desc**$(\underline{A}_{\mathcal{E}}, R)$, the category of descent objects for $\underline{A}_{\mathcal{E}}$ over R, where a morphism between two descent objects $(c_i, \alpha_{ij})_I \to (d_i, \beta_{ij})_I$ is defined to be a family $(f_i : c_i \to d_i)_I$ making the below square commute

$$\begin{array}{ccc} {\pi_i}^* c_i & \xrightarrow{\alpha_{ij}} & {\pi_j}^* c_j \\ {\scriptstyle {\pi_i}^* f_i}\downarrow & & \downarrow{\scriptstyle {\pi_j}^* f_j} \\ {\pi_i}^* d_i & \xrightarrow[\beta_{ij}]{} & {\pi_j}^* c_j \end{array}$$

Finally, there is a clear canonical functor $A^U \to$ **Desc**$(\underline{A}_{\mathcal{E}}, R)$, defined by the action $c \mapsto {\pi_i}^* c$. We will then say that $\underline{A}_{\mathcal{E}}$ *descends along* R when this canonical map is an equivalence. Furthermore, for a site $(\mathcal{E}, J)$, we will say the $\mathcal{E}$-indexed $\underline{A}_{\mathcal{E}}$ a *stack* if it descends along every R of J .

Remark 2.26. We convention **Des**$(\underline{A}_{\mathcal{E}}, \emptyset) \cong 1$.

Remark 2.27. We note that the parallel between descent objects and matching families is rather precise: given a basis K and a family $\{f_i : d_i \to c\}$ of $K(c)$, a matching family for $F : \mathcal{C}^{op} \to$ **Set** is a list of elements $x_i \in F(d_i)$ with $F\pi_i(x_i) = F\pi_j(x_j)$.

Example 2.28. Let $\mathcal{C}$ be a category with pullbacks and identify it with its canonical indexing over itself (cf. 2.23). Given a morphism $i \overset{\alpha}{\to} j$ of $\mathcal{C}$ consider the diagram below

$$i \times_j i \times_j i \; \substack{\xrightarrow{\pi_{01}} \\ \xrightarrow{\pi_{12}} \\ \xrightarrow[\pi_{02}]{}} \; i \times_j i \; \substack{\xrightarrow{\pi_0} \\ \xleftarrow{\Delta_i} \\ \xrightarrow[\pi_1]{}} \; i \xrightarrow{\alpha} j$$

An object of descent in $\mathcal{C}$ along α consists of an arrow $c \to i$ in $\mathcal{C}$ and an isomorphism $\theta : {\pi_0}^* c \cong {\pi_1}^* c$ satisfying

$${\Delta_i}^*(\theta) = 1 \quad {\pi_{12}}^*(\theta) \circ {\pi_{01}}^*(\theta) = {\pi_{02}}^*(\theta)$$

Remark 2.29. Given a locale X and a geometric morphism $\mathsf{Sh}(X) \xrightarrow{f} \mathcal{E}$ form a diagram as above,

$$\mathsf{Sh}(X) \times_{\mathcal{E}} Sh(X) \times_{\mathcal{E}} \mathsf{Sh}(X) \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \mathsf{Sh}(X) \times_{\mathcal{E}} \mathsf{Sh}(X) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathsf{Sh}(X) \xrightarrow{\;\; f \;\;} \mathcal{E}$$

As localic topoi are stable under pullbacks (cf., e.g., [18, 2.1]), we have locales Y and Z with $\mathsf{Sh}(X) \times_{\mathcal{E}} \mathsf{Sh}(X) \simeq \mathsf{Sh}(Y)$ and $\mathsf{Sh}(X) \times_{\mathcal{E}} \mathsf{Sh}(X) \times_{\mathcal{E}} \mathsf{Sh}(X) \simeq \mathsf{Sh}(Z)$. Furthermore, since Loc is 2-reflective in $\mathfrak{Top}/\mathcal{S}$, we can guarantee $Z \cong Y \times_X Y$. In short, the diagram above is reflected to the localic groupoid

$$Y \times_X Y \; \substack{\xrightarrow{d_{01}} \\ \xrightarrow{d_{02}} \\ \xrightarrow[d_{12}]{}} \; Y \; \substack{\xrightarrow{d_0} \\ \xleftarrow{i} \\ \xrightarrow[d_1]{}} \; X \qquad (\star)$$

Notice now that $Des(Sh(X)_\bullet, f)$ will have as objects pairs $(E \in \mathsf{Sh}(X), \theta : d_0^* E \xrightarrow{\cong} d_1^* x)$ satisfying the cocycle identities $i^*\theta = 1_E \quad {d_{12}}^*\theta . {d_{01}}^*\theta = {d_{02}}^*\theta$, that is to say, $Des(Sh(X)_\bullet, f)$ is the category of equivariant sheaves for the groupoid in $(\star)$.

The above remark is of historical significance, as the celebrated Joyal-Tierney equivariant sheaf representation for Grothendieck topoi was obtained using this insight. Namely, given a Grothendieck topos $\mathcal{E}$ they used Diaconescu's cover to obtain an open surjection $Sh(X) \to \mathcal{E}$ and then proved that open surjections are *effective descent*, which, in this context, means that $Des(Sh(X)_\bullet, f) \simeq \mathcal{E}$ and so, by the previous discussion, we have $BG \simeq \mathcal{E}$ for some localic groupoid G. We make note of it below.

Theorem 2.30 (Joyal-Tierney Sheaf Representation)**.** *Every Grothendieck topos is equivalent to the classifying topos of some localic groupoid.*

Following, we have the historical[7] result.

Theorem 2.31 (Bénabou-Roubaud)**.** *Let $\underline{A}_{\mathcal{E}}$ be an $\mathcal{E}$-indexed category satisfying the Beck-Chevalley condition. For any $\alpha : i \to j$ we have **Desc**$(\underline{A}_{\mathcal{E}}, \{\alpha\})$ equivalent to the coalgebras given by the comonad induced by $\alpha^* \dashv \Pi_\alpha$. Therefore, $\underline{A}_{\mathcal{E}}$ descends along α iff α^* is comonadic.*

Proof: We begin by reminding the reader of the Beck-Chevally condition. We say that an indexed category $\underline{A}_{\mathcal{E}}$ satisfies it every morphism $\alpha : i \to j$ of $\mathcal{E}$ has a right adjoint $\alpha^* \dashv \Pi_\alpha$ such that, for every pullback square as the one on the left,

$$\begin{array}{ccc} i & \xrightarrow{\alpha} & j \\ {\scriptstyle\gamma}\downarrow & \lrcorner & \downarrow{\scriptstyle\beta} \\ l & \xrightarrow[\delta]{} & k \end{array} \qquad\qquad \begin{array}{ccc} A^j & \xrightarrow{\alpha^*} & A^i \\ {\scriptstyle\Pi_\beta}\downarrow & \cong & \downarrow{\scriptstyle\Pi_\gamma} \\ A^k & \xrightarrow[\delta^*]{} & A^l \end{array}$$

[7] The below theorem seems to be one of the first that used the, now common, Beck-Chevalley condition.

The map $\delta^*\Pi_\beta \to \Pi_\gamma\alpha^*$, coming from the adjunct of $\gamma^*\delta^*\Pi_\beta \cong \alpha^*\beta^*\Pi_\beta \xrightarrow{\alpha^*\eta} \alpha^*$, is an isomorphism.

Now, given the descent object $\theta : {\pi_0}^*c \cong {\pi_1}^*c$ note that θ corresponds bijectively to some $c \to \Pi_{\pi_0}{\pi_1}^*c$ and, as $\Pi_{\pi_0}{\pi_1}^* \cong \alpha^*\Pi_\alpha$ by Beck-Chevally, we obtain a morphism $\overline{\theta} : c \to \alpha^*\Pi_\alpha c$. We now prove that this action is a coalgebra iff the initial morphism $\theta : {\pi_0}^*c \cong {\pi_1}^*c$ satisfies the cocycle conditions. First, we see that $\Delta^*(\theta)$ corresponds to the composition

$$c \xrightarrow{\overline{\theta}} \alpha^*\Pi_\alpha c \cong \Pi_{\pi_0}{\pi_1}^*c \xrightarrow{\eta^\Delta} \Pi_{\pi_0}\Pi_\Delta\Delta^*{\pi_1}^*(c) \cong \Pi_{\pi_0\Delta}(\pi_1\Delta)^*c \cong c$$

with η^Δ the unity of $\Delta^* \dashv \Pi_\Delta$. Note now that, by Beck-Chevalley, the diagram below commutes

$$\begin{array}{ccc} \alpha^*\Pi_\alpha c & \xrightarrow{\eta^\alpha} & a \\ \cong\downarrow & & \downarrow\cong \\ \Pi_{\pi_0}{\pi_1}^*c & \longrightarrow & \Pi_{\pi_0}\Pi_\Delta\Delta^*{\pi_1}^*(c) \end{array}$$

Hence, $\Delta^*\theta = 1$ if $\eta.\overline{\theta} = 1$. Analogously, we have to $\pi_{12}^*(\theta)\pi_{01}^*(\theta)$ correspond to

$$c \xrightarrow{\overline{\theta}} \alpha^*\Pi_\alpha c \xrightarrow{\alpha^*\Pi_\alpha(\overline{\theta})} \alpha^*\Pi_\alpha\alpha^*\Pi_\alpha c$$

and that $\pi_{02}^*(\theta)$ corresponds to

$$c \xrightarrow{\overline{\theta}} \alpha^*\Pi_\alpha c \xrightarrow{\mu_c} \alpha^*\Pi_\alpha\alpha^*\Pi_\alpha c$$

With μ the comonad multiplication. Hence, θ is associative if the arrow $\overline{\theta}$ is also associative. □

Remark 2.32. The condition on α in the lemma 2.31 is satisfied, for example, for the $\mathcal{S}$-indexed given by $i \mapsto \mathcal{S}/F(i)$, for $F : \mathcal{S} \to \mathcal{C}$ preserving pullbacks and $\mathcal{C}$ locally Cartesian. In particular, for any $\mathcal{S}$-topos $\mathcal{E} \to \mathcal{S}$, the $\mathcal{S}$-indexing corresponding to $\mathcal{E}$ satisfies the conditions of the lemma.

In this work, we are be primarily interested in stacks for the coherent topology – the Grothendieck topology given by the sieves that contain finite jointly epimorphic families. Conveniently, stacks for this topology have a good description when indexing along a pretopos.

Lemma 2.33. *If $\mathcal{E}$ is a pretopos, $\underline{A}_{\mathcal{E}}$ will be a stack for coherent topology iff we have*

- $A^0 \cong 1$
- *For every pair $i, j \in \mathcal{E}$, $A^{i+j} \cong A^i \times A^j$.*
- *For every epimorphism $\alpha : i \to j$, $\underline{A}_{\mathcal{E}}$ descends along α.*

Proof: If $\underline{A}_{\mathcal{E}}$ is a stack for the coherent topology, note that since the empty family covers 0 and we made the convention **Des**$(\underline{A}_{\mathcal{E}}, \emptyset) \cong 1$ we concluded $A^0 \cong 1$. Next, given pair i, j let $P = \{i \to i+j, j \to i+j\}$. As pretopoi have disjoint products, the diagram below is a fiber product

$$\begin{array}{ccc} 0 & \xrightarrow{a} & i \\ {\scriptstyle b}\downarrow & \lrcorner & \downarrow \\ j & \longrightarrow & i+j \end{array}$$

So, using $A^0 \cong 1$, given pair of objects $x \in A^i$ and $y \in A^j$ we have $a^*(x) = b^*(y)$. It is easy to verify that the identity will satisfy the cocycle conditions and, therefore, we conclude **Desc**$(\underline{A}_{\mathcal{E}}, P) \cong A^i \times A^j$. Now, since P is jointly surjective, we get $A^{i+j} \cong A^i \times A^j$. Finally, the last condition is trivially satisfied.

Conversely, given coverage $X = \{f_i : U_i \to V\}_n$ we can factor it into the families $Y = \{g_i : U_i \to \coprod_n U_i\}_n$ and $Z = \{f : \coprod_n U_i \twoheadrightarrow V\}$. By hypothesis, we have an equivalence $\Phi : A^{\coprod_n U_i} \to \prod_n A^{U_i} : \Psi$. Note that an element of **Desc**$(\underline{A}_{\mathcal{E}}, Z)$ will be a pair $(x \in A^{\coprod_n U_i}, \alpha : k^*x \to k^*x)$ with $\Delta^*\alpha = 1$. Define an arrow **Desc**$(\underline{A}_{\mathcal{E}}, Z) \to$ **Desc**$(\underline{A}_{\mathcal{E}}, X)$ by the action $(x, \alpha) \mapsto (\Psi(x), \alpha_{ij})$, with α_{ij} the obvious map. It is routine to show that this action is well defined (i.e., the family satisfies the cocycle conditions) and induces a functor. Next, define **Desc**$(\underline{A}_{\mathcal{E}}, X) \to$ **Desc**$(\underline{A}_{\mathcal{E}}, Z)$ by $((x_i)_n, \alpha_{ij}) \mapsto (\Phi((x_i)_n), \alpha)$ with α the map induced by α_{ij}. We then verify that **Desc**$(\underline{A}_{\mathcal{E}}, Z) \simeq$ **Desc**$(\underline{A}_{\mathcal{E}}, X)$. By hypothesis, we have **Desc**$(\underline{A}_{\mathcal{E}}, Z) \simeq A^V$. The result follows. □

Now, we are in a position to prove the result below, which will be fundamental in Awodey's and Breiner's sheaves representations.

Proposition 2.34. *Given a pretopos $\mathcal{P}$ its $\mathcal{P}$-canonical indexing, $\underline{\mathcal{P}}$, defined in the example* 2.23*, is a stack for the coherent topology.*

Proof: It is enough to verify that the conditions of the lemma 2.33 are satisfied. We clearly have $\mathcal{P}/0 \cong 1$ and $\mathcal{P}/(i+j) \cong \mathcal{P}/i \times \mathcal{P}/j$. For the final condition, given the epimorphism $\alpha : i \to j$, it suffices, by the 2.31 theorem, to prove that α^* will be comonadic and for that we use Beck's (co)monadicity. Since $\mathcal{P}$ is locally closed Cartesian, α^* has a right adjoint. Furthermore, $\mathcal{P}$ has all equalizers which α^*, by virtue of having a left adjoint, preserves. So all that remains is to show α^* conservative and for that we use the classic result that guarantees that the base change $\alpha^* : \mathcal{P}/j \to \mathcal{P}/i$ in a regular category is conservative iff α is a regular epimorphism (cf., e.g., [11, A, 1.3.2, 1.3.4]) and remind the reader that every epimorphism is regular in a pretopos ([*loc. cit.*, A, 1.4.9]). □

2.4. A Sheaf representation for topoi

We can now show a result of Steve Awodey's thesis, [2], which improves the first sheaf representation for topoi we obtained.

Lemma 2.35. *Every strict small stack over the coherent topology is equivalent to some sheaf over the coherent topology.*

Proof: Let $P : \mathcal{E}^{op} \to \mathfrak{CAT}$ be a stack and R a sieve of the coherent topology, the latter generated by a jointly surjective family $\{\alpha_n : A_n \to I\}_n$. Let us show that the inclusion $R \rightarrowtail \mathcal{Y}I$ induces an equivalence $Hom(R, P) \cong Hom(\mathcal{Y}I, P)$. Consider the induced arrow $y\alpha : \coprod_n \mathcal{Y}A_n \to \mathcal{Y}I$ and take its regular factorization, which we obtain as the coequalizer of the kernel pair of $\coprod_n \mathcal{Y}A_n \to \mathcal{Y}I$,

$$\begin{array}{ccccc} \coprod_n \mathcal{Y}A_n \times_{\mathcal{Y}I} \coprod_n \mathcal{Y}A_n & \rightrightarrows & \coprod_n \mathcal{Y}A_n & \xrightarrow{q} & R \\ & & & \searrow^{y\alpha} & \downarrow r \\ & & & & \mathcal{Y}I \end{array}$$

Applying the functor $Hom(-, P)$ and letting $yA := \coprod_n \mathcal{Y}A_n$ we can form the diagram below

$$\begin{array}{ccccccc} Hom(R, P) & \xrightarrow{q^*} & Hom(yA, P) & \underset{q_0}{\overset{q_1}{\rightrightarrows}} & Hom(yA \times_{\mathcal{Y}I} yA, P) & \Rrightarrow & P(yA \times_{\mathcal{Y}I} yA \times_{\mathcal{Y}I} yA) \\ \uparrow r^* & \nearrow_{(y\alpha)^*} & & & & & \\ Hom(\mathcal{Y}I, P) & & & & & & \end{array}$$

Note that $q^* : Hom(R, P) \rightarrowtail Hom(yA, P)$ is the equalizer of the pair that follows it. Now, using lemma 2.33 and letting $A := \coprod_n A_n$, we have

$$Hom(yA, P) = Hom\left(\coprod_n \mathcal{Y}A_n, P\right) \cong \prod_n Hom(\mathcal{Y}A_n, P) \cong \prod_n P(A_n) \cong P\left(\coprod_n A_n\right) = P(A)$$

We also have

$$Hom\left(yA \times_{yI} yA, P\right) = Hom\left(\coprod_n \mathcal{Y}A_n \times_{\mathcal{Y}I} \coprod_n \mathcal{Y}A_n, P\right) \cong Hom\left(\coprod_{n,m} \mathcal{Y}\left(A_n \times_I A_m\right), P\right)$$

$$\cong \prod_{n,m} Hom\left(\mathcal{Y}\left(A_n \times_I A_m\right), P\right) \cong \prod_{n,m} P\left(A_n \times_I A_m\right)$$

$$\cong P\left(\coprod_{n,m}(A_n \times_I A_m)\right) \cong P\left(\coprod_n A_n \times_I \coprod_n A_n\right) = P(A \times_I A)$$

Analogously we show $P(A \times_I A \times_I A) \cong Hom\left(yA \times_{\mathcal{Y}I} yA \times_{\mathcal{Y}I} yA, P\right)$. Putting $\alpha : \coprod_n A_n \to I$ the arrow induced by the family, we may form the diagram below, with u an equivalence,

$$\begin{array}{ccccccc} des(\alpha) & \rightarrowtail & P(A) & \rightrightarrows & P(A \times_I A) & \rightrightarrows\!\!\!\!\!\!\to & P(A \times_I A \times_I A) \\ {\scriptstyle u}\uparrow & \nearrow_{\alpha^*} & & & & & \\ P(I) & & & & & & \end{array}$$

Using the equivalences above, we guarantee $(y\alpha)^*$ a pseudoequalizer. Therefore, there is, by the universal property, a morphism $s : Hom(R, P) \to Hom(\mathcal{Y}I, P)$ and a natural isomorphism $\theta : (y\alpha)^* s \Rightarrow q^*$ such that $q_0\theta = q_1\theta$. As q^* is monic, we can conclude s faithful. Furthermore, note that $(y\alpha)^* sr^* \cong q^* r^* = (rq)^* = \alpha^*$ and so, because u is an equivalence and, therefore, $(\mathcal{Y}\alpha)^*$ is monic, we conclude $sr^* = 1$, that is, s is essentially subjective. Finally, all that remains is to prove the full s. Given $x, y : R \to P$ and $f : sx \to sy$, using $q_0\theta = q_1\theta$ it is routine to check that for $f' := \theta_y \circ (\mathcal{Y}\alpha)^* f \circ \theta_x^{-1}$ we have $q_0 f' = q_1 f'$, thus there is $h : x \to y \in Hom(R, P)$ with $q^* h = f'$. Observe now that $\theta_y \circ (\mathcal{Y}\alpha)^* s(h) = q^*(h) \circ \theta_x$, and so

$$(\mathcal{Y}\alpha)^* s(h) = \theta_y^{-1} \circ q^*(h) \circ \theta_x = \theta_y^{-1} \circ f' \circ \theta_x = \theta_y^{-1} \circ \theta_y \circ (\mathcal{Y}\alpha)^* f \circ \theta_x^{-1} \circ \theta_x = (\mathcal{Y}\alpha)^* f$$

As $(\mathcal{Y}\alpha)^*$ is faithful, it follows that s is full. We conclude s an equivalence and, as $sr^* = 1$, we get our result. □

Corollary 2.36. *An essentially small stack in the coherent topology is equivalent to a sheaf in the coherent topology.*

In particular, by lemma 2.34, given a small $\mathcal{E}$ topos their externalization $\underline{\mathcal{E}}$ is equivalent to a sheaf of small categories, which we shall denote by $\overline{\mathcal{E}} : \mathcal{E}^{op} \to \mathbf{Cat}$.

Definition 2.37. We will say *local* a topos whose terminal object is projective and indecomposable. Note that under the standard semantics of a topos these properties equate to, respectively, "$M \vDash \varphi \vee \psi$ iff $M \vDash \varphi$ or $M \vDash \psi$" and "$M \vDash \exists x)(\varphi(x))$ iff $M \vDash \varphi(c)$, for some c".

Lemma 2.38. *Given a topos $\mathcal{E}$ and a point* **Set** $\xrightarrow{x}$ *$Sh_{Coh}(\mathcal{E})$, the stalk $x^*(\overline{\mathcal{E}})$ is a local topos.*

Proof: First, let us show $x^*(\overline{\mathcal{E}})$ a topos. We recall that the stalks of a sheaf F are given by

$$x^*(F) \cong \varinjlim_{(c,z)\in\int\mathcal{Y}^{\text{Sh}}} Fc$$

for $\mathcal{Y}^{\text{Sh}}$ the sheafification of $\mathcal{Y}$. In particular, $x^*(\overline{\mathcal{E}}) \cong \varinjlim_{\int\mathcal{Y}^{\text{Sh}}} \overline{E}(c) \cong \varinjlim_{\int\mathcal{Y}^{\text{Sh}}} \mathcal{E}/c$. Notice that $\int\mathcal{Y}^{\text{Sh}}$ is filtered, since $\mathcal{Y}^{\text{Sh}}$ is left exact. Furthermore, by the fundamental theorem of topoi all slices $\mathcal{E}/c$ are topoi. Finally, as filtered colimits of topoi exist (cf, e.g., [21, 2.5]) we have $x^*(\overline{\mathcal{E}})$ a topos. Now, let us show that $\overline{\mathcal{E}}$ is local. Given $p, q \in Sub_{x^*\overline{E}}(1)$ with $p \vee q = 1$, there are $(I_p, y_p), (I_q, y_q) \in \int\mathcal{Y}^{\text{Sh}}$ and $p' \rightarrowtail 1$ in $\mathcal{E}/I_p$ and $q' \rightarrowtail 1$ in $\mathcal{E}/I_q$ projecting on, respectively, p and q. Because $\int\mathcal{Y}^{\text{Sh}}$ is filtered, there is (I, y) with arrows $(I, y) \to (I_p, y_p)$ and $(I, y) \to (I_q, y_q)$. By restricting p' and q' along those arrows we get $p'', q'' \rightarrowtail 1$ in $\mathcal{E}/I$. Since $p \vee q = 1$ in colimit, there is $h : (J, z) \to (I, y)$ with $h^*(p'' \vee q'') = 1$. Note then that given $a \rightarrowtail 1$ and $b \rightarrowtail 1$ in $\mathcal{E}/c$ with $a \vee b = 1$ we have $a + b \to 1$ epic, and so, letting $m : p + q \rightarrowtail 1$ we have either $m^*(p) = 1$ or $m^*(q) = 1$. Therefore, there exists some $k : (K, w) \to (J, z)$ with either $k^*h^*(p'') = 1$ or $k^*h^*(q'') = 1$. Observe that, passing to the colimit we have that $k^*h^*(p'')$ projects to to p and that $k^*h^*(q'')$ projects to q. Therefore, we have shown 1 indecomposable in $x^*(\overline{\mathcal{E}})$. The proof that 1 is projective is perfectly analogous, so we omit it and direct the curious reader to [2, V,2.1]. □

Finally, we may obtain the second sheaf representation.

Theorem 2.39 (Awodey)**.** *Given a small topos $\mathcal{E}$, there exists a space $A_{\mathcal{E}}$ and a sheaf $F_{\mathcal{E}}$ over $A_{\mathcal{E}}$ so that*

i) For every point $P \in A_{\mathcal{E}}$, the stalk $(F_{\mathcal{E}})_P$ is a local topos;

ii) $\mathcal{E}$ is isomorphic to the global sections of $F_{\mathcal{E}}$, i.e., $\mathcal{E} \simeq \Gamma(F_{\mathcal{E}})$

iii) There is a conservative logical morphism $\mathcal{E} \rightarrowtail \prod_{P\in \textbf{Spec}(\mathcal{E})}(F_{\mathcal{E}})_P$.

Namely, we can put $A_{\mathcal{E}} = X_{Sh_{Coh}(\mathcal{E})}$ and $F_{\mathcal{E}} := \varphi^*(\overline{\mathcal{E}})$, where $Sh_{Coh}(\mathcal{E}) \xrightarrow{\varphi^*} Sh(X_{Sh_{Coh}(\mathcal{E})})$ comes from the 2.14 theorem, which we can apply by the Deligne's Theorem.

We can further improve the result in the Boolean case.

Theorem 2.40. *We say a category to be well-pointed iff the global sections functor $Hom(1, -)$ is faithful. Given small Boolean topos $\mathcal{E}$ and putting $F_{\mathcal{E}}$ as above, we can guarantee that every fiber $(F_{\mathcal{E}})_P$ is well-pointed.*

Proof: See [2, V, 2.4]. □

Finally, we mention an interesting logical property of local topos

Theorem 2.41. *Intuitionistic logic is complete for local topoi models.*

Proof: See [*loc. cit*, V, 3.3]. □

3. Schemes

In this final section, we follow closely Spencer Breiner's PhD thesis, [4]. As the final instance of a sheaf representations, we present logical affine schemes. As the name implies, they shall be the objects we associate first-order theories to in an effort to mirror the relationship between affine scheme and commutative rings to our logical context.

Finally, we include a short exposition of the application of the methods in this chapter, describing the isotropy group of a topos by means of definable sets.

3.1. The Method of Diagrams

Here we show an adaptation of the classical technique of Robinson's diagrams to the context of categorical logic. The objects defined here will be fundamental in what follows, as the stalks of the aforementioned affine schemes will be described in terms of these diagrams. For the reader's convenience, we begin by recalling the definition of a diagram of a model.

Definition 3.1. Let $\mathbb{T}$ be a theory in the signature Σ. The *Robinson diagram* of a $\mathbb{T}$-model M is an extension $\mathbb{T} \subseteq \mathcal{D}(M)$ over the signature Σ_M, obtained by adding to the original Σ a constant c_m for each each $m \in M$, and whose sequences are $\mathbb{T} \cup \{\top \vdash \varphi(c_a) : a \in \varphi^M\}$.

Notoriously, the diagram of a given model classifies the homomorphisms under it.

Theorem 3.2. *Given $\mathbb{T}$-model M, homomorphisms $h : M \to N$ are in bijection with extensions of N to a $\mathcal{D}(M)$-model.*

Proof: Given extension $N' \supseteq N$ let $h(m) := c_m^{N'}$ and given homomorphism $h : M \to N$ let $c_m^{N'} := h(m)$. It is trivial to check that these actions are well defined. □

Remarkably, we may obtain the classifying pretopos of the Robinson diagram of a given model by means of a filtered colimit. We start by recalling a standard category theory result,

Lemma 3.3. *Given a preotpos $\mathcal{P}$ and a filtered diagram $J^{op} \xrightarrow{D} \mathcal{P}$, the colimitcategory category $\varinjlim_{j \in J^{op}} \mathcal{P}/Dj$ is also a pretopos.*

Definition 3.4. Given a model $\mathcal{P} \xrightarrow{M} \mathbf{Set}$, as M preserves finite limits we have $\int M$ – its category of elements – to be a filtered category (cf. [24, VIII, 6.4]). Define then the *categorical diagram* of M as the colimit of $\mathcal{P}$ by the projection $\int M^{op} \to \mathcal{P}$, that is,

$$\mathbf{Diag}(M) := \varinjlim_{(A,x) \in \int M} \mathcal{P}/A$$

We then obtain the following result.

Theorem 3.5. *Let $\mathcal{P}$ be the classifying pretopos of the theory $\mathbb{T}$ and $\mathcal{P} \xrightarrow{M}$ **Set** a model. **Diag**(M) is the classifying pretopos of the Robinson diagram of the model (corresponding to) M.*

We first prove the lemma below.

Lemma 3.6. *Let $\mathcal{P}$ be a pretopos and φ an element of $\mathcal{P}$. The slice $\mathcal{P}/\varphi$ classifies the elements definable by φ, that is to say, given a pretopos $\mathcal{Q}$ we have the equivalence*

$$\mathbf{PTopos}(\mathcal{P}/\varphi, \mathcal{Q}) \cong \bigsqcup_{M:\mathcal{P}\to\mathcal{Q}} \mathcal{Q}(1, M\varphi)$$

Proof: We clarify that the objects of the category on the right are pairs of the form $(M : \mathcal{P} \to \mathcal{Q}, a : 1 \to M\varphi)$ and that the morphisms $(M, a) \to (N, b)$ are transformations $\alpha : M \Rightarrow N$ with $\alpha_\varphi(a) = b$.

Now, given some functor $M : \mathcal{P}/\varphi \to \mathcal{Q}$ we map it to the pair $(M\varphi^*, M(\Delta_\varphi))$, where $\varphi^* : \mathcal{P} \to \mathcal{P}/\varphi$ is given by the action $A \mapsto (\pi : A \times \varphi \to \varphi)$ and Δ_φ the diagonal $\Delta_\varphi : \varphi \to \varphi \times \varphi$. The action on the arrows is taken as the obvious one.

Reciprocally, given a pair (M, a) we define $\overline{M} : \mathcal{P}/\varphi \to \mathcal{Q}$ by sending $x : A \to \varphi$ to the pullback of Mx along a. It is then routine to verify that these actions are well defined and are mutually inverse. For more details, see [4, 2.3.1]. □

Proof of the Theorem (sketch): By the universal property, a functor **Diag**$(M) \to \mathcal{Q}$ factors as a cocone $(\mathcal{P}/\varphi \to \mathcal{Q})_{\varphi \in \int M}$ which, by the lemma, is equivalent to a family of pairs $(N_\varphi : \mathcal{P} \to \mathcal{Q}, a : 1 \to N_\varphi\varphi)_{\varphi \in \int M}$. Why, the functors N_φ correspond to models $\overline{N}_\varphi \in \mathbb{T}$-**Mod**$(\mathcal{Q})$. Therefore, a functor **Diag**$(M) \to \mathcal{Q}$ determines, for each pair of the form $(\varphi \in \mathcal{P}, x \in M\varphi)$, a $\mathbb{T}$-model N_φ in $\mathcal{Q}$ and a constant $c_x : 1 \to N_\varphi\varphi$. Considering the colimit N of the models N_φ we obtain a model $N \in \mathbb{T}$-**Mod**$(\mathcal{Q})$. Therefore, a functor **Diag**$(M) \to \mathcal{Q}$ is equivalent to a model $\mathbb{T}$-**Mod**$(\mathcal{Q})$ associated of constants c_x for each $x \in M\varphi$, that is, a $D(M)$-model.

For more details, see [4, 2.4.3]. □

Proposition 3.7. *Given a model $\mathcal{P} \xrightarrow{M}$ **Set**, its diagram **Diag**(M) is a local pretopos, that is to say, the terminal object of **Diag**(M) is both projective and indecomposable.*

Proof: See [4, 2.4.8]. □

Finally, we mention the relation, from the categorical point of view, of definable sets to the diagram of our model.

Definition 3.8. Given a model M, say some subset $S \subseteq MA$ to be *definable* iff there is a formula $\varphi \rightarrowtail A \times B$ and $b \in MB$ so that $S = \{x \in MA : M \vDash \varphi(x, b)\}$. We let $Def(M)$ denote the category of definable sets for some model M.

Proposition 3.9. *The quotient-conservative factorization of the model **Diag**$(M) \to$ **Set** is given by*

$$\begin{array}{ccc} \mathbf{Diag}(M) & \longrightarrow & \mathbf{Set} \\ & \searrow\!\!\!\!\searrow \quad \nearrow & \\ & Def(M) & \end{array}$$

Proof: See [4, 2.4.4]. □

4. Logical Schemes

We can now present logical schemes, but first let us recall some facts. Given a pretopos $\mathcal{P}$, its canonical indexing over itself, $\overline{\mathcal{P}}$, defined in the example 2.23, is a stack for the coherent topology and therefore (cf. corollary 2.36) is equivalent to a sheaf in the coherent topology $\underline{\mathcal{P}} \in \mathrm{Sh}_{Coh}(\mathcal{P})$ which in turn, by theorem 2.18, is equivalent to an equivariant sheaf $\mathcal{O}_\mathcal{E} \in BG_{\mathrm{Sh}_{Coh}}(\mathcal{P})$. Equivalently, as every pretopos $\mathcal{P}$ classifies a theory $\mathbb{T}$, we may equivalently describe $\mathcal{O}_\mathcal{P}$ as an equivariant sheaf over $BG_\mathbb{T}$, by proposition 2.7. We call $\mathcal{O}_\mathcal{P}$ as the *structural sheaf of* $\mathcal{P}$. We will give below an explicit description of this sheaf as an object of $BG_\mathbb{T}$, but first we remind the reader of the description of the groupoid associated to our theory $\mathbb{T}$.

Remark 4.1. During the following sections, we will work with pretopoi associated with theories they classify, that is, pairs $\mathcal{P}$ and $\mathbb{T}$ with $\mathcal{P} \simeq PTop(\mathbf{Syn}(\mathbb{T}))$, where $PTop(-)$ is the pretopoi completion. Since the functor $\mathbf{Syn}(\mathbb{T}) \to PTop(\mathbf{Syn}(\mathbb{T}))$ is full and conservative, we usually identify $\mathbf{Syn}(\mathbb{T})$ with its image in $\mathcal{P}$ and so treat (α-equivalence classes) of coherent formulas as objects of $\mathcal{P}$.

Definition 4.2 (bis)**.** Let $X_{\mathbb{T}}$ the space of models whose underlying sets are elements of $\kappa := \max(|\Sigma|, \omega)$ and whose topology is given by the basics

$$B_{\varphi(a)} := \{M \in X_{\mathbb{T}} : a \in M\varphi\}$$

For φ a coherent formula with n free variables and $a = (a_i)_n \in \kappa^n$. Next, let $G_{\mathbb{T}}$ the groupoid of isomorphisms between the models of $X_{\mathbb{T}}$, whose topology is the coarsest making the domain and codomain maps $d_0, d_1 : G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ continuous and containing, for each type A and pair $a, b \in \kappa$, the sets below as opens

$$V_{A, a \mapsto b} := \{f : M \xrightarrow{\cong} N : a \in MA, f_A(a) = b\}$$

We now obtain a description of $\mathcal{O}_{\mathcal{P}}$.

Theorem 4.3. *Let $\mathcal{P}$ the classifying pretopos of the theory $\mathbb{T}$. We have $\mathcal{O}_{\mathcal{P}}(B_{\varphi(a)}) \simeq \mathcal{P}/\varphi$. In special, $\Gamma(\mathcal{O}_{\mathcal{P}}) \simeq \mathcal{P}$.*

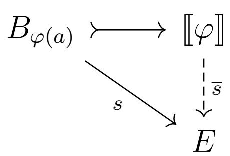
Proof: Recall that we established the equivalence $BG_{\mathbb{T}} \simeq \mathrm{Sh}_{Coe}(\mathbf{Syn}(\mathbb{T}))$ showing (proposition 2.6) that the equivariant sheaves $([\![\varphi]\!], \theta)$ generate the topos $BG_{\mathbb{T}}$, where

$$[\![\varphi]\!]_{X_{\mathbb{T}}} = \left\{\langle M, \overline{a}\rangle : M \in X_{\mathbb{T}}, \overline{a} \in [\![\varphi]\!]^M\right\} \xrightarrow{\pi} X_{\mathbb{T}} \quad \text{and} \quad \theta(f : M \to N, (M, a)) = (N, f(a))$$

Therefore, we may conclude[8] that $BG_{\mathbb{T}} \simeq \mathrm{Sh}_{Coe}(\mathcal{P})$ sends $([\![\varphi]\!], \theta)$ to the representable $\mathcal{Y}\varphi$. Thus,

$$Hom([\![\varphi]\!], \mathcal{O}_{\mathcal{P}}) \simeq Hom(\mathcal{Y}\varphi, \overline{\mathcal{P}})$$

Now, given equivariant étale space (E, μ) of $BG_{\mathbb{T}}$ and section $s : B_{\varphi(a)} \to E$, we claim that there is an unique extension



Indeed, given $(M, b) \in [\![\varphi]\!]$ it is routine to construct an isomorphism $f : M \Rightarrow N$ sending

[8]Recall that, for $\mathcal{P} \simeq PTop(\mathbf{Syn}(\mathbb{T}))$ we have an equivalence $\mathrm{Sh}_{Coe}(\mathbf{Syn}(\mathbb{T})) \simeq \mathrm{Sh}_{Coe}(PTop(\mathbf{Syn}(\mathbb{T})))$ given by lifting the inclusion $\mathbf{Syn}(\mathbb{T}) \hookrightarrow Ptop(\mathbf{Syn}(\mathbb{T}))$

the sequence b to a. As $\overline{s}$ must be equivariant, we shall have

$$\begin{aligned}\overline{s}((M,b)) &= \theta(f^{-1}f, \overline{s}((M,b))) \\ &= \theta(f^{-1}, \theta(f, \overline{s}((M,b)))) \\ &= \theta(f^{-1}, \overline{s}(f(M,b))) \\ &= \theta(f^{-1}, \overline{s}(N,a)) \\ &= \theta(f^{-1}, s(N))\end{aligned}$$

Since θ equivariant, it is clear the above expression does not depend on the choice of f and that it uniquely determines the action of $\overline{s}$. In short, for an equivariant sheaf F, we have $F(B_{\varphi(a)}) \cong Hom([\![\varphi]\!], F)$. Finally, we may obtain

$$\mathcal{O}_{\mathcal{P}}(B_{\varphi(a)}) \cong Hom([\![\varphi]\!], \mathcal{O}_{\mathcal{P}}) \simeq Hom(\mathcal{Y}\varphi, \overline{\mathcal{P}}) \simeq \overline{\mathcal{P}}(\varphi) = \mathcal{P}/\varphi$$

In special, $\Gamma(\mathcal{O}_{\mathcal{P}}) = Hom([\![1]\!], \mathcal{O}_{\mathcal{P}}) = \simeq Hom(\mathcal{Y}1, \overline{\mathcal{P}}) \simeq \overline{\mathcal{P}}/1 \simeq \mathcal{P}$. □

Theorem 4.4. *Let $\mathcal{P}$ the classifying pretopos of $\mathbb{T}$. Given a model $M \in X_{\mathbb{T}}$, we have $(\mathcal{O}_{\mathcal{P}})_M \simeq$ **Diag**(M)*

Proof: Check [4, 3.2.4]. □

Overall, we have obtained the following representation.

Theorem 4.5 (Breiner)**.** *Let $\mathcal{P}$ be the classifying pretopos of the $\mathbb{T}$ theory. Setting $G_{\mathbb{T}} \rightrightarrows X_{\mathbb{T}}$ the groupoid and $\mathcal{O}_{\mathcal{P}} \in BG_{\mathbb{T}}$ the equivariant sheaf defined above, we have*

i) For every point $M \in X_{\mathbb{T}}$, the stalk $(\mathcal{O}_{\mathcal{P}})_M$ is a local pretopos;
ii) $\mathcal{P}$ is isomorphic to the global sections of $\mathcal{O}_{\mathcal{P}}$, i.e., $\mathcal{P} \simeq \Gamma(\mathcal{O}_{\mathcal{P}})$
iii) There is a conservative morphism of pretopos $\mathcal{P} \rightarrowtail \prod_{M \in X_{\mathbb{T}}} (\mathcal{O}_{\mathcal{P}})_M$.

We are then led to the following definition.

Definition 4.6. For all pretopos $\mathcal{P}$, we denote by **Spec**$(\mathcal{P})$ the pair $(\mathcal{P}, \mathcal{O}_{\mathcal{P}})$, and refer to it as the *logical affine scheme associated with $\mathcal{P}$.*

4.1. Axiomatized Spaces

Continuing our translation of Algebraic Geometry techniques to the logical context, we adapt the following classical definitions (cf., e.g., [10, 1]) to our environment.

Definition 4.7. An *axiomatized space* is a pair $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$, where $\mathcal{G}$ is a topological groupoid and $\mathcal{O}_{\mathcal{G}}$ an equivariant sheaf of pretopoi over $\mathcal{G}$. We say $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ *locally axiomatized* iff, for all $x \in \mathcal{G}_0$, we have the stalk $(\mathcal{O}_{\mathcal{G}})_x$ a local pretopos. A morphism between axiomatized spaces $(\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \to (\mathcal{F}, \mathcal{O}_{\mathcal{F}})$ is a pair consisting of a morphism of groupoids $f : \mathcal{G} \to \mathcal{F}$ and a pretopoi morphism $\phi : \mathcal{O}_{\mathcal{F}} \to f_* \mathcal{O}_{\mathcal{G}}$ int $B\mathcal{F}$. A morphism between locally axiomatized spaces $(f, \phi) : (\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \to (\mathcal{F}, \mathcal{O}_{\mathcal{F}})$ is a morphism between axiomatized spaces such that the transposed $\phi^\sharp : f^* \mathcal{O}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{F}}$ preserves the maximal ideal (of the subobjects of 1) of each stalk of $f^* \mathcal{O}_{\mathcal{G}}$.

Let us show that the affine scheme functor **Spec**$(-)$ is a map from **PTopos** to the locally axiomatized spaces.

Theorem 4.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be the pretopoi classifying, respectively, $\mathbb{T}_0$ and $\mathbb{T}_1$. A pretopoi morphism $\mathcal{P} \to \mathcal{Q}$ induces a groupoid morphism $f : G_{\mathbb{T}_1} \to G_{\mathbb{T}_0}$ and a pretopoi morphism $\phi : \mathcal{O}_{\mathcal{P}} \to f_* \mathcal{O}_{\mathcal{Q}}$ such that the fibers of the transposed map $\phi^\sharp : f^* \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{Q}}$ are conservative*[9].

Proof: Using the equivalences

$$\mathbb{T}_0\text{-}\mathbf{Mod}(\mathbf{Set}) \simeq \mathbf{PTopos}(\mathcal{P}, \mathbf{Set}) \quad \mathbb{T}_1\text{-}\mathbf{Mod}(\mathbf{Set}) \simeq \mathbf{PTopos}(\mathcal{Q}, \mathbf{Set})$$

it becomes clear that we have a morphism $f : G_{\mathbb{T}_1} \to G_{\mathbb{T}_0}$ induced by postcomposition. Furthermore, recall that the $\mathbb{T}_1$-**Mod**(**Set**) $\to$ **PTopos**$(\mathcal{Q}, \mathbf{Set})$ half of the equivalence is given by sending a model M to the functor whose action on **Syn**$(\mathbb{T}_1)$ is $\psi \mapsto [\![\psi]\!]^M$. Now, given $\varphi \in \mathcal{P}$ we will have

$$\begin{aligned} f^{-1}(B_{\varphi(k)}) &= \{M \in X_{\mathbb{T}_1} : k \in [\![\varphi]\!]^{fM}\} \\ &\cong \{\mathcal{Q} \xrightarrow{\overline{M}} \mathbf{Set} : k \in (\overline{M}F)(\varphi)\} \\ &= \{\mathcal{Q} \xrightarrow{\overline{M}} \mathbf{Set} : k \in \overline{M}(F(\varphi))\} \\ &\cong \{M \in X_{\mathbb{T}_1} : k \in [\![F\varphi]\!]^M\} \\ &\cong B_{(F\varphi)(k)} \end{aligned}$$

And so $f_0 : X_{\mathbb{T}_1} \to X_{\mathbb{T}_0}$ is continuous. Analogously, we show that f_1 is continuous and therefore we have f in fact a groupoid morphism. Next, ϕ is defined in the basics as the obvious

$$\mathcal{O}_{\mathcal{P}}(B_\varphi(k)) \simeq \mathcal{P}/\varphi \to \mathcal{Q}/F\varphi \simeq \mathcal{O}_{\mathcal{Q}}(B_{F\varphi(k)})$$

Finally, remembering that the stalks of $\mathcal{O}_{\mathcal{P}}$ are diagrams (cf. theorem 4.4) and that the inverse image f^* preserves stalks, it is easy to see that the stalks of the transpose $\phi^\sharp : f^* \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{Q}}$ are of the form **Diag**$(FM) \to$ **Diag**(M). Now, recalling that a morphism between pretopoi is conservative iff it is injective in subobjects (cf., e.g.,

[9] and, therefore, preserve the maximal ideal of $f^* \mathcal{O}_{\mathcal{G}}$

[4, 2.2.1]) and that **Diag**(M) is the pretopos classifier of $D(M)$, we may conclude conservative the morphism in question for FM is a reduct of the model M, cf. [*loc. cit.*, 3.3.1]. □

We are now in a position to define our version of scheme, but first we need to talk about subopens and covers.

Definition 4.9. Given axiomatized space $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ and subgroupoid $U \subseteq \mathcal{G}$, the *axiomatized subspace associated* to U is defined by restricting $\mathcal{O}_{\mathcal{G}}$ to U_0, with equivariant action inherited from $U_1 \subseteq \mathcal{G}_1$. We shall identify a subgroup with its associated subspace. A subspace $U \subseteq \mathcal{G}$ is said to be open if $U_0 \subseteq G_0$ and $U_1 \subseteq G_1$ are both open. Finally, we say a family of open subspaces $\{U_i\}_I$ to be an *open cover* for $\mathcal{G}$ iff any $\alpha : x \to y \in \mathcal{G}_1$ there is a sequence $\beta_i : z_i \to z_{i+1} \in (U_{i_n})_1$, $i \in [0, n)$, with $z_0 = x$ and $z_n = 1$ and $\alpha = \beta_n\beta_{n-1}\cdots\beta_1\beta_0$. Note that, in particular, we will have $\bigcup_I (U_i)_0 = \mathcal{G}_0$.

Definition 4.10 (Schemes)**.** A *logical scheme* is a locally axiomatized space $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ that admits an open cover $\{U_i\}_I$ such that there is pretopoi $\mathcal{P}_i$ with $U_i \simeq$ **Spec**$(\mathcal{P}_i)$. Let **LogSch** the full subcategory of **AxSp** whose objects are logical scheme.

We mention in passing the following interesting result, which allows us to treat logical schemes as descent objects.

Proposition 4.11. *If $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ is a logical scheme and $\{U_i\}_I$ an open cover of it then the canonical geometric morphism $i : \prod_I BU_i \to B\mathcal{G}$ is an open surjection. In particular, since open surjective morphism are effective descent (cf. [19, 2, thm. 1]), we have $B\mathcal{G} \simeq Des(i)$, where the descent here is as in example 2.28.*

Proof: Check [4, 3.3.5]. □

Next, we establish that schemes are stable and that logical schemes are stable for basics.

Lemma 4.12. *The open subspace $B_{\varphi(a)} \subseteq$ **Spec**$(\mathcal{P})$ is affine, with $B_{\varphi(k)} \simeq$ **Spec**$(\mathcal{P}/\varphi)$. The open subspace $U \subseteq \mathcal{G}$ of a scheme $\mathcal{G}$ is a scheme.*

Proof: See [4, 3.4.1] □

We now show that it is possible to glue our schemes.

Remark 4.13. Below, to ease reading (and writing), we let $\mathcal{G}^1$ and $\mathcal{G}^0$ denote, respectively, the morphisms and objects of a groupoid $\mathcal{G}$.

Lemma 4.14 (Gluing)**.** *Let $(\mathcal{G}_i, \mathcal{O}_{\mathcal{G}_i})_I$ a family of axiomatized spaces, $(U_{ij})_I$ a family of open subspaces $U_{ij} \subseteq \mathcal{G}_i$ and $\varphi_{ij} : U_{ij} \to U_{ji}$ a family of isomorphisms of* ***AxSp*** *such that these data satisfy $U_{ii} = \mathcal{G}_i$, $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ e*

$$\varphi_{ii} = 1_{\mathcal{G}_i} \quad \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \quad (\star)$$

Then there is axiomatized space $(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$, open cover $\{U_i\}_I$ of $\mathcal{G}$ and isomorphisms of axiomatiate spaces $\varphi_i : \mathcal{G}_i \to U_i$ such that $\varphi_i(U_{ij}) = U_i \cap U_j$ e $\varphi_{ij} = \varphi_j^{-1} \circ \varphi_i$. Furthermore, given space $(\mathcal{F}, \mathcal{O}_{\mathcal{F}})$, the morphisms between axiomatized spaces $f : \mathcal{G} \to \mathcal{F}$ are in bijection with families $f_i : \mathcal{G}_i \to \mathcal{F}$ satisfying $f_j \circ \varphi_{ij} = f_i$.

Proof: For $k \in \{0, 1\}$, let $\mathcal{G}^k := \coprod_I (\mathcal{G}_i^k)/\sim$ where $(x, i) \sim (y, j)$ iff $x \in U_{ij}^k$, $y \in U_{ji}^k$ and $\varphi_{ij}^k(x) = y$, note that the equations $(\star)$ imply $\sim$ an equivalence. Now, let $\varphi_i^k : \mathcal{G}_i^k \to \mathcal{G}^k$ the inclusion maps, the topology in $\mathcal{G}^k$ being defined by "$V \subseteq \mathcal{G}^k$ is open iff $(\varphi_i^k)^{-1}(V) \subseteq \mathcal{G}_i^k$ for all $i \in I$". The codomain map $d_0 : \mathcal{G}^1 \to \mathcal{G}^0$ is given by, representing as $[-]$ the equivalence classes, $[f : x \to y] \mapsto [x]$. Since $d_0\varphi_{ij}^1 = \varphi_{ij}^0 d_0$ we conclude the action well-defined. Furthermore, given open $V \subseteq \mathcal{G}_i^0$ we have

$$(\varphi_i^1)^{-1}(d_0^{-1}(V)) = (d_0\varphi_i^1)^{-1}(V) = (\varphi_i^0 d_0)^{-1}(V) = d_0^{-1}((\varphi_i^0)^{-1}(V))$$

And so d_0 is continuous. Analogously, we show the other groupoid maps o well defined and continuous. Now, setting $U_i := \varphi_i(\mathcal{G}_i)$ observe that $(\varphi_i^k)^{-1}U_j = U_{ij}^k$ and so it is clear that the family is an open cover for $\mathcal{G}$. Following, given open $W \subseteq \mathcal{G}_i^k$ we have $(\varphi_j^k)^{-1}(\varphi_i(W)) = (\varphi_{ij}^k)^{-1}(W \cap U_{ij})$, thus $\varphi_i^k : \mathcal{G}_i \to U_i$ are homeomorphisms[10]. Finally, we define the sheaf $\mathcal{O}_{\mathcal{G}}$ gluing the sheaves $\mathcal{O}_{\mathcal{G}_i}$ as usual. Its equivariant action is defined stalkwise: given $f : x \to y \in \mathcal{G}^1$ there is $i \in I$ and $\overline{f} : \overline{x} \to \overline{y}$ with $\varphi_i^1(\overline{f}) = f$, define then

$$((d_0)^*\mathcal{O}_{\mathcal{G}})_f \cong (\mathcal{O}_{\mathcal{G}})_{d_0(f)} = (\mathcal{O}_{\mathcal{G}})_x \cong (\mathcal{O}_{\mathcal{G}_i})_{\overline{x}} \to (\mathcal{O}_{\mathcal{G}_i})_{\overline{y}} \cong ((d_1)^*\mathcal{O}_{\mathcal{G}})_f$$

Next, given the family f_i as in the thereom's statement; define, for $x \in U_i$ and $k \in \{0, 1\}$, maps $f^k(\varphi_i(x)) = f_i^k(x)$. The action is obviously well defined. Forgetting the groupoid structure, the statement is enough information to glue the sheaves and so we get a morphism of sheaves $\mathcal{O}_{\mathcal{F}} \to f_*\mathcal{O}_{\mathcal{G}}$ and, as equivariance may be checked stalkwise, we clearly have it a equivariant action. Furthermore, as we have $(\mathcal{O}_{\mathcal{G}})_{\varphi_i(x)} \cong (\mathcal{O}_{\mathcal{G}_i})_x$ we note that the maps will preserve the maximal ideal.

Finally, given $f : \mathcal{G} \to \mathcal{F}$ let $f_i := f \circ \varphi_i$. It is routine then to verify that this family will satisfy the required condition. □

[10] It's routine to show that an homeomorphism of groupoids lifts to an isomorphisms of axiomatized spaces

We now obtain a fundamental result.

Theorem 4.15. *We have the following adjunction*

$$\mathbf{PTopos}^{op} \underset{\Gamma}{\overset{\textbf{\textit{Spec}}}{\rightleftarrows}} \textbf{\textit{LogSch}} \qquad (\top)$$

In special, $\mathbf{PTopos}(\mathcal{E}, \mathcal{F}) \simeq \textbf{\textit{LogSch}}(\textbf{\textit{Spec}}(\mathcal{F}), \textbf{\textit{Spec}}(\mathcal{E}))$.

Proof (sketch): Given $\mathcal{G}$, fix an affine coverage $\{\textbf{Spec}(\mathcal{P}_i)\}$. Note that global sections induce a family $s_i : \Gamma(\mathcal{G}) \to \mathcal{E}_i$ which, by the theorem 4.8, lifts to $\overline{s_i} : \textbf{Spec}(\mathcal{E}_i) \to Spec(\Gamma(\mathcal{G}))$. We then define the counit $\eta : \mathcal{G} \to \textbf{Spec}(\Gamma(\mathcal{G}))$ by $\eta(x) := \overline{s_i}(x)$ to $x \in \textbf{Spec}(\ cP_i)$. We define the structural map $\varphi^\sharp : \eta^*\mathcal{O}_\Gamma(\mathcal{G}) \to \mathcal{O}_\mathcal{G}$ using that, by 4.11, we have $B\mathcal{G} \simeq Des(J)$ and so inherit the maps $s_i^\sharp : s_i^*\mathcal{O}_\Gamma(\mathcal{G}) \to \mathcal{O}_{\textbf{Spec}(\mathcal{P}_i)}$. Next, the unit is given by the equivalence $\mathcal{P} \simeq \Gamma(\textbf{Spec}(\mathcal{P}))$.

For verification of triangle diagrams, check [4, 3.5.1] □

Theorem 4.16. *The category of logical schemes admits finite limits, which are calculated as colimits in the category of pretopoi.*

Proof: Check [4, 3.5.4]. The idea of the proof is clear: use the 4.15 theorem to directly obtain the limits of affine scheme as pretopoi colimits and, in the general case, glue along an affine basis, using lemma 4.14. □

4.2. Isotropy Group

We now present an application of our logical scheme methods. Here we consider the theory of "Crossed Topoi", developed by Jonathon Funk, Pieter Hofstra and Benjamin Steinberg in [16]. The objects of which are the topos-theoretical analogues to the crossed modules of homological algebra. Their work explores (and defines!) the isotropy group of a topos, an object that can be used to induce canonical crossed structure. Our interest here, however, in the logical description of this group obtained by Breiner, who characterizes it by means of definable automorphisms.

Definition 4.17. Given a topos $\mathcal{E}$, the *isotropy functor* is the map $\mathcal{Z} : \mathcal{E}^{op} \to \mathbf{Grp}$ that associates to each $a \in \mathcal{E}$ the group of automorphisms of the projection $\mathcal{E}/a \to \mathcal{E}$. In Funk *et al* work, it is verified that $\mathcal{Z}$ preserves colimits and is, therefore, representable. We call *isotropy group* the group $Z \in \mathbf{Grp}(\mathcal{E})$ that represents our functor $\mathcal{Z}$.

A special case of interest is presented below.

Proposition 4.18. *For $\mathcal{P}$ a pretopos, the isotropy group of $Sh_{Coh}(\mathcal{P})$ is given by*

$$Z(A) := \mathit{Aut}(A^*) = \left\{ \alpha \,\middle|\, \mathcal{P} \underset{}{\overset{\cong\,\alpha}{\rightrightarrows}} \mathcal{P}/A \right\}$$

Proof: [4, 4.3.3] □

We now introduce the logical objects that will describe our isotropy group.

Definition 4.19. Given pretopos $\mathcal{P}$ and model $M : \mathcal{P} \to$ **Set**, we say that an automorphism $\alpha : M \cong M$ is definable iff for every type B there is object A_B, element $x_0 \in A_B^M$ and formula $\sigma(x, y, z)$ such that

$$\alpha(a) = b \iff M \vDash \sigma(a, b, x_0)$$

Let A a type. Given a family of formulas $\{xyz : \sigma_B\}_B$, with $x, y : B$ and $z : A$, we say it a family of A-definable automorphisms iff for every model M and $a \in A^M$, the formulas $\{\sigma_B(x, y, a)\}_B$ define an automorphism of M.

Let M a model. Given a family of formulas $\{\sigma_B(x, y, a_B)\}_B$ in the Robinson diagram of M, with $x, y : B$ and a_B a constant, we say it a family of M-definable automorphisms iff for every homomorphism $h : M \to N$ the formulas $\{\sigma_B(x, y, h(a_B))\}_B$ define an automorphism of N.

Proposition 4.20. *Given pretopos $\mathcal{P}$ and $A \in \mathcal{P}$, the isotropy group $Z(A)$ is isomorphic to the family of A-definable automorphisms. Furthermore, given model $M : \mathcal{P} \to$ **Set**, the fiber Z_M of the isotropy group is isomorphic to the family of M-definable automorphisms.*

Proof: See [4, 4.3.7, 4.3.9]. □

A. Appendix

A.1. Syntax

We define below the syntax of our language. We work in a typed language, as usual in categorical logic. For a deeper treatment of this and the two following subsections, we recommend [12, D] or [9]

Definition A.1. A *signature* Σ is

- A set Σ_t, whose elements we call *types*;
- For each sequence $A_1, A_2, \cdots, A_n$ of types, a (possibly empty) set $(\Sigma_R)_{A_1,A_2,\cdots,A_n}$ of *relations*. We write $R \leq A_1 A_2 \cdots A_n$ to denote that R is in $(\Sigma_R)_{A_1,A_2,\cdots,A_n}$. We allow $n = 0$ and, in this case, call R a propositional variable;

- For each sequence $A_1, A_2, \cdots, A_n, B$ of types, a (possibly empty) set $(\Sigma_f)_{A_1,A_2,\cdots,A_n,B}$ of *functions*. We write $f : A_1A_2\cdots A_n \to B$ to denote that f is in $(\Sigma_f)_{A_1,A_2,\cdots,A_n,B}$. Similarly, we allow $n = 0$ and in this case we call f a constant and write $f : B$;

Example A.2. We give two examples of signatures,

a) Ring theory is generally specified in the signature with a single type $\star$, pair of constants $0, 1 : \star$, functions $+, * : \star\star \to \star$ and $(-)^{-1} : \star \to \star$ and no relation.
b) The theory of vector spaces is usually specified in a signature with two types, one for scalars and one for vectors.

We can now define a language over our signature.

Definition A.3. Given a signature Σ, *the terms over* Σ are a family of sets $(\mathbf{Term}(\Sigma)_A)_{A\in\Sigma_t}$ defined recursively by

i) For each natural number i and type A of Σ, we have a *variable* x_i^A in $\mathbf{Term}(\Sigma)_A$. Generally, we will omit the superscript from the variables.
ii) For each constant $c : A$, we have $c \in \mathbf{Term}(\Sigma)_A$;
iii) For each function $f : A_1A_2\cdots A_n \to B$ and sequence $t_i \in \mathbf{Term}(\Sigma)_{A_i}$ for $i \leq n$, we have $f(t_1, t_2, \cdots, t_n) \in \mathbf{Term}(\Sigma)_B$.

We usually write $t \in \mathbf{Term}(\Sigma)_A$ as $t : A$.

Next, the set of *formulas over* Σ, denoted by $\mathbf{Form}(\Sigma)$, is recursively defined by

i) $\top, \bot \in \mathbf{Form}(\Sigma)$;
ii) For each pair of terms $t, s \in \mathbf{Term}(\Sigma)$ with $t, s : A$, we have $t = s \in \mathbf{Form}(\Sigma)$;
iii) For each relation $R \leq A_1A_2\cdots A_n$ and sequence $t_i \in \mathbf{Term}(\Sigma)_{A_i}$ for $i \leq n$, we have $R(t_1, t_2, \cdots, t_n) \in\ form.$
iv) For each pair $\varphi, \psi \in \mathbf{Form}(\Sigma)$ we have $\varphi \vee \psi$ and $\varphi \wedge \psi$ in $\mathbf{Form}(\Sigma)$;
v) For each $\varphi \in \mathbf{Form}(\Sigma)$, i natural and type A we have $\exists x_i^A \varphi \in \mathbf{Form}(\Sigma)$;
vi) For each pair $\varphi, \psi \in \mathbf{Form}(\Sigma)$ we have $\varphi \to \psi$ and $\neg\varphi$ in $\mathbf{Form}(\Sigma)$.
vii) For each $\varphi \in \mathbf{Form}(\Sigma)$, i natural and type A we have $\forall x_i^A \varphi \in \mathbf{Form}(\Sigma)$;

The subset of $\mathbf{Form}(\Sigma)$ closed for conditions $i)$ to $v)$ will be called the set of *coherent formulas of* Σ, denoted by $\mathbf{Form}_{Coh}(\Sigma)$. In this work, we shall concern ourselves mostly to this kind of formula.

Formulas support a notion similar to that of type, but to describe it we need to define the concept of a free variable. Intuitively, we will say free any variable that is not quantified.

Definition A.4. Given a Σ signature, recursively define the VL function in $\mathbf{Term}(\Sigma)$ by setting $VL(x_i) = x_i$ and $VL(f(t_1, t_2, \cdots, t_n)) = \bigcup_n VL(t_i)$. Next, define VL in $\mathbf{Form}(\Sigma)$ recursively by

i) $VL(\top) = VL(\bot) = \emptyset$;
ii) $VL(t = s) = VL(t) \cup VL(S)$;
iii) $VL(R(t_1, t_2, \cdots, t_n)) = \bigcup_n VL(t_i)$;
iv) $VL(\varphi \wedge \psi) = VL(\varphi \wedge \psi) = VL(\varphi \to \psi) = VL(\varphi) \cup V(\psi)$;
v) $VL(\exists x_i^A \varphi) = VL(\forall x_i^A \varphi) = VL(\varphi) \setminus \{x_i^A\}$;
vi) $VL(\neg\varphi) = VL(\varphi)$.

The *free variables* of a formula φ are the elements of $VL(\varphi)$.

We can now define contextualized formulas. A *context* is a finite list $\overline{x} = x_1 x_2 \cdots x_n$ of distinct variables. The case $n = 0$ is allowed, and we denote this *empty context* by []. The *type of a context* is the list (possibly with repetitions) of the types of variables that occur in the context in question, in order of appearance. We will say a $\overline{x}$ context suitable for a φ formula if all free variables of φ occur in $\overline{x}$. A *contextualized formula* is a pair $\overline{x}.\varphi$, where $\overline{x}$ is a context suitable for the formula φ.

Remark A.5. A variable can be free in a formula even though it has non-free instances, as in $\varphi = (\exists x_1(x_1 = x_2)) \vee (x_1 = x_1)$, where the first instance of x_1 is not-free, but $VL(\varphi) = \{x_1, x_2\}$.

We can now define theories.

Definition A.6. A *sequent* is a formal expression $(\varphi \vdash_{\overline{x}} \psi)$, with φ and ψ formulas and $\overline{x}$ a context suitable to both. A *theory* is a (possibly empty) set of formulas.

A.2. Categorical Semantics

We recall here the standard technique by which we may interpret logical expressions inside a category.

Definition A.7. Let Σ be a signature and $\mathcal{P}$ a pretopos. A Σ-structure M in $\mathcal{P}$ consists of the following data,

- For each A type of Σ, one object MA of $\mathcal{P}$. We extend this definition by setting $M\overline{A} := \prod_n MA_i$ for product types $\overline{A} = A_1 A_2 \cdots A_n$;
- For each relation $R \leq \overline{A}$, a subobject $MR \leq M\overline{A}$ of $\mathcal{P}$;
- For each function $f : \overline{A} \to B$, a map $Mf : M\overline{A} \to MB$ in $\mathcal{P}$;

A *homomorphism* $M \xrightarrow{h} N$ between two Σ-structures M and N in $\mathcal{P}$ is a family of morphisms $MA \xrightarrow{h_A} NA$ of $\mathcal{P}$ for each type A such that

i) For every relation $R \leq \overline{A}$, there is a morphism $MR \to NR$ making the diagram below commute

$$\begin{array}{ccc} MR & \rightarrowtail & M(A_1 \cdots A_n) \\ \downarrow & & \downarrow {\scriptstyle (h_{A_i})_n} \\ NR & \rightarrowtail & N(A_1 \cdots A_n) \end{array}$$

ii) For every function $\overline{A} \xrightarrow{f} B$ we have $h_B \circ Mf = Nf \circ (\prod_n h_{A_i})$

We define Σ-$Str(\mathcal{P})$ as the category of homorphisms between Σ-structures.

Next, we can extend the interpretations of a Σ-structure to the rest of our language.

Definition A.8. Let $\mathcal{P}$ be a pretopos and $M \in \Sigma\text{-}Str(\mathcal{P})$. Given a term $t : B$ and a context $\overline{x}$ of type $A_1 A_2 \cdots A_n$ such that all variables present in t occur in $\overline{x}$, we define the arrow $[\![\overline{x}.t]\!]^M : M\overline{A} \to MB$ recursively by

- If t is a variable, then it is one of the x_i and set $[\![\overline{x}.t]\!]^M$ as the i-th projection of $M(\overline{A})$;
- If t is of the form $f(t_1, \cdots, t_n)$ then $[\![\overline{x}.t]\!]^M := Mf \circ ([\![\overline{x}.t_i]\!]^M)_n$

Analogously, we recursively define the interpretations of coherent formulas. Assuming that in all the cases below $\overline{x}$ is suitable context for the formula in question, we define that

- $[\![\overline{x}.\top]\!]^M$ and $[\![\overline{x}.\bot]\!]^M$ are, respectively, the maximum and minimum object of $Sub_{\mathcal{P}}(M\overline{A})$;
- For each pair of terms $t, s \in$ **Term**(Σ) with $t, s : B$ for some type B, $[\![\overline{x}.(t = s)]\!]^M$ is the equalizer of the pair $[\![\overline{x}.t]\!]^M, [\![\overline{x}.s]\!]^M : \; M\overline{A} \rightrightarrows MB$;
- For each relation $R \leq B_1 B_2 \cdots B_n$ and sequence of terms $t_i \in$ **Term**$(\Sigma)_{B_i}$, $[\![\overline{x}.R\overline{t}]\!]^M$ is defined by the pullback below

$$\begin{array}{ccc} [\![\overline{x}.R(t_1, t_2, \cdots, t_n)]\!]^M & \longrightarrow & MR \\ \downarrow & \lrcorner & \downarrow \\ M\overline{A} & \xrightarrow[([\![\overline{x}.t_i]\!]^M)_n]{} & M\overline{B} \end{array}$$

- For each pair $\varphi, \psi \in$ **Form**(Σ), we set

$$[\![\overline{x}.(\varphi \vee \psi)]\!]^M := [\![\overline{x}.\varphi]\!]^M \vee [\![\overline{x}.\psi]\!]^M \qquad [\![\overline{x}.(\varphi \wedge \psi)]\!]^M := [\![\overline{x}.\varphi]\!]^M \wedge [\![\overline{x}.\psi]\!]^M$$

  Where each operation is taken in $Sub_{\mathcal{P}}(M\overline{A})$;
- For each formula $\varphi \in$ **Form**(Σ) and variable y of type B, we set $[\![\overline{x}.\exists y \varphi]\!]^M$ as the image of the composition

$$[\![\overline{x}y.\varphi]\!]^M \rightarrowtail M(A_1 A_2 \cdots A_n B) \xrightarrow{\pi} M(A_1 \overline{A})$$

Given a coherent sequent $\sigma = (\varphi \vdash_{\overline{x}} \psi)$, we say that M satisfies σ – in symbols, $M \vDash \sigma$ – if, and only if, $[\![\overline{x}.\varphi]\!]^M \leq [\![\overline{x}.\psi]\!]^M$ in $Sub(M\overline{A})$. When M satisfies all sequences of a theory $\mathbb{T}$ we say it is a *model of* $\mathbb{T}$. We denote by $\mathbb{T}$-**Mod**$(\mathcal{P})$ the full subcategory of Σ-$Str(\mathcal{P})$ whose objects are models of $\mathbb{T}$.

Remark A.9. Of course, if we reduce our considerations to fragments of our language, we don't need all the categorical structure that a pretopos has. A trivial example: if we are only interested in formulas generated by conjunctions of atomic sentences then a category with finite limits will suffice

Lemma A.10. *Pretopoi functors preserve validation. Explicitly, let* $F : \mathcal{P} \to \mathcal{Q}$ *in* $\mathbf{PTopos}$ *and* $M \in \mathbb{T}\text{-}Mod(\mathcal{P})$. *Define* $FM \in \Sigma\text{-}Str(\mathcal{Q})$ *by setting* $(FM)A := F(MA)$, $(FM)R := F(MR)$ *and* $(FM)f := F(Mf)$. *We have* $FM \in \mathbb{T}\text{-}Mod(\mathcal{Q})$. *Furthermore, given a sequent* σ, *if* $M \vDash \sigma$ *then* $FM \vDash \sigma$, *with the reciprocal being valid if* T *is conservative.*

Proof: A trivial induction. For the final condition, just note that $M \vDash (\varphi \vdash \psi)$ iff $[\![\varphi \wedge \psi]\!] \cong [\![\varphi]\!]$. □

Example A.11. We mention two examples of interpretations of theories.

1. A common example of interpreting theories on a topos is given by sheaves of rings, which the reader may recognize as **CRing** $- Mod(\text{Sh}(X))$, with **CRing** the theory of commutative rings.
2. Another common example is that Lie groups, which we recognize as **Group** $-$ Mod(**SmthMan**), with **Group** the theory of groups and **SmthMan** a category of smooth manifolds. Note that **SmthMan** is not a pretopos, but it still has sufficient structure[11] to interpret group theory, cf. observation A.9.

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