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A new kind of Mersenne numbers: the tricomplex Mersenne sequence

Fernando Soares Carvalho [1], **Paula M. M. C. Catarino** [2], **Helena Campos** [2], **and Eudes Antonio Costa** [1]

[1]*Department of Mathematics, Federal University of Tocantins, Brazil.*
[2]*Department of Mathematics, University of Trás-os-Montes and Alto Douro, Portugal.*

E-mails: fscarvalho@uft.edu.br, pcatarin@utad.pt, hcampos@utad.pt, eudes@uft.edu.br

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Abstract

In this paper, a new family of arithmetic sequences associated with Mersenne numbers is introduced, and their properties are explored. The classical Mersenne sequence $\{M_n\}_{n\geq 0}$ is defined by the initial values $M_0 = 0$ and $M_1 = 1$, with the recurrence relation $M_n = 3M_{n-1} - 2M_{n-2}$ for $n \geq 2$. This concept is extended by defining the tricomplex Mersenne numbers within the tricomplex ring $(\mathbb{T}, +, \times)$. A homogeneous recurrence relation for these new numbers is presented, and their various mathematical properties are studied. The tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$ is defined by the initial values TM_0^* and TM_1^*, as well as the same Mersenne recurrence relation. Key formulas, such as the Binet formula, are derived, and fundamental identities like those of Tagiuri-Vajda, d'Ocagne, and Catalan are explored. In addition, generating functions are presented, and novel identities related to the Mersenne numbers are discovered through this extended framework. Several properties of the tricomplex Mersenne sequence are also established, and results involving sums of terms are provided. These findings contribute to a deeper understanding of generalized Mersenne-type sequences and their potential applications.

Keywords: Binet formula; Mersenne-type sequences; Tricomplex Mersenne sequences; Generating function.

MSC2020: 11B83, 11B37

1 Introduction

A considerable number of researchers have concentrated their efforts on sequences of positive integers defined recursively. In this paper, an extension of this type is presented, taking as a basis one of these integer sequences, namely the Mersenne sequence, which is also related to another example known as the Mersenne–Lucas sequence.

The foundation of the study presented in this article is the sequence of Mersenne numbers. The Mersenne numbers are defined by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2}, \quad \text{for } n \geq 2,$$

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with initial values $M_0 = 0$ and $M_1 = 1$. This recurrence provides the basis for studying their properties and applications in number theory, generating the sequence $\{M_n\}_{n\geq 0} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, \dots\}$, which is cataloged as sequence $A000225$ in the OEIS [1]. Let $n \geq 0$ be an integer; the same recurrence relation for the Mersenne–Lucas numbers $\{m_n\}_{n\geq 0}$ is $m_n = 3m_{n-1} - 2m_{n-2}$, with the initial terms $m_0 = 2$ and $m_1 = 3$. Using this recurrence relation, the first few terms of the Mersenne–Lucas sequence can be generated by $\{m_n\}_{n\geq 0} = \{2, 3, 5, 9, 17, 33, 65, 129, \dots\}$, the sequence $A000051$ in the OEIS [1]. Note that the same recurrence is satisfied by both M_n and m_n, and numerous identities are fulfilled by both numbers, many of which were discovered centuries ago. Herein, employing the simplified notation, let us denote $\{M_n^*\}_{n\geq 0}$ by M_n or m_n. Thus, the sequence $\{M_n^*\}_{n\geq 1}$ of Mersenne-type is satisfied by $M_{n+2}^* = 3M_{n+1}^* - 2M_n^*$; $n \geq 0$, with initial terms M_0^* and M_1^*. If $M_0^* = M_0 = 0$ and $M_1^* = M_1 = 1$, the ordinary Mersenne numbers are obtained; conversely, when $M_0^* = m_0 = 2$ and $M_1^* = m_1 = 3$, the Mersenne–Lucas numbers are derived. You can check in [2], [3], and others. The Mersenne numbers hold significant importance in the exploration of prime numbers; numerous studies on this subject are extensively cited in the literature: see [4], [5], [6], [7], and [8].

In this paper, a new extension of the Mersenne-type sequence is introduced; it shall be called the tricomplex Mersenne sequence. Unlike other variations, this new extension depends on two real vectors with three coordinates used in a linear recurrence relation. In accordance with the findings of [9], [10], [11] and [12], we consider the tricomplex number ring, denoted by $\mathbb{T}$, as the set of ordered triples of real numbers, represented by the ordered real triple (x, y, z).

Now, in the real tricomplex ring $(\mathbb{T}, +, \times)$, we define the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$.

Definition 1.1. For all natural numbers n, the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$ is given by:

$$TM_n^* = (M_n^*, M_{n+1}^*, M_{n+2}^*) ,$$

where $\{M_n^*\}_{n\geq 1}$ is the Mersenne-type sequence.

For instance, we have $TM_0 = (0, 1, 3)$, $TM_1 = (1, 3, 7)$, and $TM_2 = (3, 7, 15)$; while $Tm_0 = (2, 3, 5)$, $Tm_1 = (3, 5, 9)$, and $Tm_2 = (5, 9, 17)$.

The aim of this work is to define the tricomplex Mersenne sequence as an extension of the Mersenne sequence and to examine some of their properties.

This article is organized as follows. In Section 2, firstly, we introduce the $\mathbb{T}$ tricomplex ring, establishing the foundation for the numerical ring being studied, with additional details available in the cited references. Additionally, we focus on the Mersenne-type sequence, highlighting essential results that support the progression of this research. In Section 3, we will study the tricomplex Mersenne numbers and their second-order vector recurrence relation. Additionally, we derive the vector Binet formulas and establish several identities associated with the sequence. Furthermore, we present the ordinary generating function and the exponential generating function for the tricomplex Mersenne sequence. In Section 4, we will explore several fundamental identities for tricomplex Mersenne, such as Tagiuri-Vajda, d'Ocagne, and their respective consequences. Finally, in Section 5, we analyze some sums of terms involving the tricomplex Mersenne numbers.

Recent studies have explored the extension of classical integer sequences within the framework of tricomplex numbers. In an early contribution, Costa et al. [13] introduced the tricomplex Fibonacci sequence and developed a new class of Fibonacci-type numbers, establishing a variety of fundamental identities. Subsequently, in [14], the authors investigated the symmetry properties of the tricomplex Repunit sequence and its relationship with the classical repunit sequence. In a related direction, Santos et al. [15] proposed a novel approach to the Horadam sequence, examining both its intrinsic properties and its tricomplex counterpart. More recently, Spreafico

et al. [16] introduced the tricomplex Leonardo numbers, a new generalization of the classical Leonardo sequence, and derived several of their fundamental properties, including recurrence relations and connections with the tricomplex Fibonacci and tricomplex Lucas numbers. These contributions provided significant motivation for the present work. In particular, they inspired our investigation of the tricomplex Mersenne numbers, introduced in Section 3, and established a conceptual framework for the analyses and results developed throughout this paper.

2 Background and preliminaries results

This section is divided into two distinct subsections. The first provides a brief introduction to the $\mathbb{T}$ tricomplex ring, laying the groundwork for the numerical ring under investigation; further details can be found in the indicated references. The second subsection revisits the Mersenne-type sequence, presenting key results that will be used in the development of this study.

2.1 The tricomplex ring

The study of rings analogous to the ring of integers, wherein arithmetic concepts can also be developed, has achieved significant interest among researchers. Notably, the ring of Gaussian integers, whose investigation traces back to Gauss's work on cubic and biquadratic reciprocity, is highlighted. This ring is particularly fascinating due to the fact that many arithmetic results in Gaussian integers are analogous to those in integers and can be illustrated geometrically. The construction and historical context of the complex number field and the tricomplex ring can be found in sources such as [9], [10], [11], [17], [18], [19], [20], [21], and [22] among others.

In mathematics, a tricomplex number is defined as an element of a number system that extends the complex numbers. Analogous to how complex numbers consist of a real part and an imaginary part, tricomplex numbers encompass a real part and two imaginary parts. The concept of tricomplex numbers was introduced by Olariu [11] and is expressed in the form $(x, y, z) = x + y\mathbf{i} + z\mathbf{j}$, where x, y, and z are real numbers, and $\mathbf{i}$ and $\mathbf{j}$ are imaginary units. The operations of addition $(+)$ and multiplication $(\times)$ are defined as follows:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \tag{1}$$

and

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_1x_2 + y_1z_2 + z_1y_2, z_1z_2 + x_1y_2 + y_1x_2, y_1y_2 + x_1z_2 + z_1x_2). \tag{2}$$

where $(x_i, y_i, z_i) \in \mathbb{T}$, for $i \in \{1, 2, 3\}$, and xy is the ordinary multiplication in $\mathbb{R}$. The multiplication rules for tricomplex numbers are detailed in Table 1 above:

Table 1: The multiplication table for tricomplex units

$\times$	1	$\mathbf{i}$	$\mathbf{j}$
1	1	$\mathbf{i}$	$\mathbf{j}$
$\mathbf{i}$	$\mathbf{i}$	$\mathbf{j}$	1
$\mathbf{j}$	$\mathbf{j}$	1	$\mathbf{i}$

According to Equation (1) the sum of the tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2) is the tricomplex number $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$. And according to Equation (2), the product of the tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2) is the tricomplex number $(x_1x_2 + y_1z_2 + z_1y_2, z_1z_2 + x_1y_2 + y_1x_2, y_1y_2 + x_1z_2 + z_1x_2)$. Hence, this product simply follows from the

products of tricomplex units given in Table 1. So, tricomplex numbers and their operations can be represented by $\mathbb{T} = (\mathbb{T}, +, \times)$. It can be checked through direct calculation that the tricomplex zero is $(0, 0, 0)$, denoted simply as 0, and the tricomplex unity is $(1, 0, 0)$, denoted simply as 1. Moreover, it can be checked that $\mathbb{T}$ is the commutative and unit ring known as a tricomplex ring $\mathbb{T}$ (see [9], [11], [12], and references).

2.2 Mersenne numbers

The algebraic, polynomial, and symmetric-function aspects of Mersenne-type sequences have attracted considerable attention in the mathematical literature, leading to a variety of theoretical developments and applications. Several studies have contributed to a deeper understanding of these structures and their underlying properties. Among these contributions, Catarino, Campos, and Vasco [2] introduced the Mersenne sequence and established a number of fundamental identities associated with it. Subsequently, Chelgham and Boussayoud [23] investigated the properties of the (k)-Mersenne–Lucas numbers, extending the scope of Mersenne-type sequences. Further advances were made by Saba and Boussayoud [24], who developed the theory of bivariate Mersenne–Lucas polynomials and examined several of their algebraic properties. More recently, Saba, Boussayoud, and Kanuri [25] explored the relationship between Mersenne–Lucas numbers and complete homogeneous symmetric functions. In addition to establishing these connections, the authors derived and analyzed a variety of identities involving Mersenne–Lucas numbers, thereby contributing to the enrichment of the theoretical framework surrounding Mersenne-type sequences.

For simplicity, we adopt the notation $\{M_n^*\}_{n\geq 0}$, and denote this sequence by either M_n or m_n. The sequence $\{M_n^*\}_{n\geq 0}$ follows a Mersenne-type

$$M_{n+2}^* = 3M_{n+1}^* - 2M_n^*, \tag{3}$$

with the initial values M_0^* and M_1^*. So, when the initial values are set as $M_0^* = M_0 = 0$ and $M_1^* = M_1 = 1$, the sequence generates the classic Mersenne numbers. On the other hand, if we take $M_0^* = m_0 = 2$ and $M_1^* = m_1 = 3$, the sequence corresponds to the Mersenne–Lucas numbers.

We emphasize that the notation $\{M_n^*\}_{n\geq 0}$ is not used here to denote the generalized Mersenne sequence (also known as the Gersenne sequence), which is conventionally represented by $\{G_n\}_{n\geq 0}$ in the literature; see, for example, [8] and the references therein.

Recall that the characteristic equation associated with the Equation (3) for the Mersenne sequence is given by $r^2 = 3r + 2$. This equation corresponds to the characteristic equation of a Horadam-type sequence (see [26] and [27]). Solving for r yields the roots of the equation. As shown in [2] and [25], the distinct roots are $r_1 = 2$ and $r_2 = 1$. Using the Binet formula offers a straightforward method to calculate the n-th Mersenne number without needing to iterate through the sequence.

Lemma 2.1. *[Binet's Formula] Let $\{M_n^*\}_{n\geq 0}$ be Mersenne-type. Then:*

$$M_n^* = 2^n + (-1)^{\xi(M^*)}, \tag{4}$$

where $\xi(M^*) = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi(M^*) = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

This formula above provides a direct expression for computing Mersenne-type numbers.

To make notation easier, we will write ξ instead of $\xi(M^*)$.

In the literature, the function $GF_{a_n}(x)$ is known as the ordinary generating function for the sequence $\{a_n\}_{n\geq 0}$, with

$$GF_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

In addition, using Equation (3), the next result presents the ordinary generating function for the Mersenne sequence.

Lemma 2.2. *[28, Corollary 6] The ordinary generating function for the Mersenne-type sequence $\{M_n^*\}_{n\geq 0}$, denoted by $GF_{M_n^*}(x)$, is given by*

$$GF_{M_n^*}(x) = \frac{M_0^* + (-1)^{\xi+1} M_1^* x}{2x^2 - 3x + 1} ,$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

In literature, the exponential generating function $E_{a_n}(x)$ of a sequence $\{a_n\}_{n\geq 0}$ is a power series of the form

$$GE_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} .$$

Combining the Equation (4) and the definition of the exponential generating function, a direct calculation gives us:

Lemma 2.3. *The exponential generating function for the Mersenne-type sequence $\{M_n^*\}_{n\geq 0}$, denoted by $GE_{M_n^*}(x)$, is given by*

$$GE_{M_n^*}(x) = \sum_{n=0}^{\infty} \frac{M_n^* x^n}{n!} = e^{2x} + (-1)^{\xi} e^{x} ,$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Using Binet's formula, Lemma 2.1, we will establish the convolution identity and Tagiuri-Vajda identity for the Mersenne-type sequence $\{M_n^*\}_{n\geq 0}$.

Proposition 2.4. *[Convolution Identity] Let $\{M_n^*\}_{n\geq 0}$ be Mersenne-type sequence. For any natural numbers k, n, we have*

$$M_k^* M_{n-1}^* + M_{k+1}^* M_n^* = 5M_{k+n-1} + (-1)^{\xi} 3\left(M_k + M_{n-1} + M_1^*\right) + 4 ,$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. By Equation (4) we get

$$\begin{aligned}
& M_k^* M_{n-1}^* + M_{k+1}^* M_n^* \\
= \; & \left(2^k + (-1)^{\xi}\right)\left(2^{n-1} + (-1)^{\xi}\right) + \left(2^{k+1} + (-1)^{\xi}\right)\left(2^n + (-1)^{\xi}\right) \\
= \; & (2^2+1)2^{k+n-1} + (-1)^{\xi}(2+1)2^k + (-1)^{\xi}(2+1)2^{n-1} + 2 \\
= \; & 5M_{k+n-1} + (-1)^{\xi} 3\left(M_k + M_{n-1} + M_1^*\right) + 4 ;
\end{aligned}$$

as required. □

Proposition 2.5. *[Tagiuri-Vajda identity] Let $\{M_n^*\}_{n\geq 0}$ be Mersenne-type sequence, and r, s, k be any non-negative integers. We have the following identity:*

$$M_{r+s}^* M_{r+k}^* - M_r^* M_{r+s+k}^* = (-1)^{\xi+1} 2^r M_s M_k ,$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. Using Equation (4) again, we obtain that

$$\begin{aligned} & M^*_{r+s}M^*_{r+k} - M^*_r M^*_{r+s+k} \\ = & \left(2^{r+s} + (-1)^\xi\right)\left(2^{r+k} + (-1)^\xi\right) - \left(2^r + (-1)^\xi\right)\left(2^{r+s+k} + (-1)^\xi\right) \\ = & (-1)^\xi\left(2^{r+s+k} - 2^{r+k} - 2^{r+s} + 2^r\right) \\ = & (-1)^{\xi+1}2^r(2^s-1)(2^k-1)\,, \end{aligned}$$

and we have the validity of the result. □

The next result is a combination of Proposition 2.5.2 in [2] and Theorem 2.8 in [25].

Lemma 2.6. *Let $\{M^*_n\}_{n\geq 1}$ be the Mersenne–type sequence, then*

$$\sum_{k=0}^{n} M^*_k \;=\; 2^{n+1} - 1 + (n+1)(-1)^\xi\,,$$

where $\xi = 1$ if $\{M^*_n\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M^*_n\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Again, using Lemma 2.1, we present the partial sums of the Mersenne-type sequence, separated by terms with odd indexes and those with even indexes.

Proposition 2.7. *Let $\{M^*_n\}_{n\geq 1}$ be the Mersenne–type sequence, then*

$$\begin{aligned} &(a) \quad \textstyle\sum_{k=0}^{n} M^*_{2k} &&= \frac{4^{n+1}-1}{3} + (-1)^\xi(n+1)\,, \\ &(b) \quad \textstyle\sum_{k=0}^{n} M^*_{2k+1} &&= 2\frac{4^{n+1}-1}{3} + (-1)^\xi(n+1)\,. \end{aligned}$$

where $\xi = 1$ if $\{M^*_n\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M^*_n\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. (a) Note that

$$\begin{aligned} \sum_{k=0}^{n} M^*_{2k} &= [2^0 + (-1)^\xi] + [2^2 + (-1)^\xi] + [2^4 + (-1)^\xi] + \cdots + [2^{2n} + (-1)^\xi] \\ &= (2^0 + 2^2 + 2^4 + \cdots + 2^{2n}) + [(-1)^\xi + (-1)^\xi + \cdots + (-1)^\xi] \\ &= \frac{4^{n+1}-1}{4-1} + (-1)^\xi(n+1)\,, \end{aligned}$$

which verifies the result.

(b) In a similar way

$$\begin{aligned} \sum_{k=0}^{n} M^*_{2k+1} &= [2^1 + (-1)^\xi] + [2^3 + (-1)^\xi] + [2^5 + (-1)^\xi] + \cdots + [2^{2n+1} + (-1)^\xi] \\ &= 2(2^0 + 2^2 + 2^4 + \cdots + 2^{2n}) + [(-1)^\xi + (-1)^\xi + \cdots + (-1)^\xi] \\ &= 2\frac{4^{n+1}-1}{3} + (-1)^\xi(n+1)\,, \end{aligned}$$

this completes the proof. □

The sequences of integers that satisfy the recurrence relation

$$H_{n+1} = pH_n - qH_{n-1}, \text{ for all } n \geq 1 , \tag{5}$$

where p and q are fixed, and H_1 and H_0 are given, are referred to as *Horadam sequences*. According to Equation (3) in [29], let us now consider the matrix of order 2 of the form

$$\mathcal{H} = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix},$$

which is associated with the *Horadam sequences*. The Gersenne sequence studied in [15] presents a generalized class of Mersenne numbers, so this generalization is essentially analogous to the generalized Horadam sequence for a second-order recurrence with $p - q = 1$. In Equation (5), by setting $p = 3$, $q = 2$, $H_0 = 0$, and $H_1 = 1$, we obtain the recurrence relation $H_{n+1} = 3H_n - 2H_{n-1}$, so the Mersenne sequence $\{M_n\}_{n\geq 0}$ is a kind of *Horadam sequence*. Thus, when the Mersenne sequence is considered, the resulting matrix is given by $\mathcal{M} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$.

The following result will prove to be of considerable utility to us and can be found in [29].

Lemma 2.8. *[29, Theorem 2.1] Given the matrix* $\mathcal{H} = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}$. *For every non negative integer* $n \geq 1$, *we have*

$$\mathcal{H}^n = \begin{bmatrix} H_{n+1} & -2H_n \\ H_n & -2H_{n-1} \end{bmatrix},$$

where $\{H_n\}_{n\geq 0}$ *is the Horadam sequences.*

In order to conclude this subsection, we will present a specific result concerning the Mersenne sequence $\{M_n\}_{n\geq 0}$.

Proposition 2.9. *For all non negative integers* r *and* s, *we obtain the recurrence relation:*

$$M_{r+s} = M_r M_{s+1} - 2M_{r-1}M_s, \quad r \geq 1, .$$

where $\{M_n\}_{n\geq 0}$ *is the Mersenne sequence.*

Proof. Applying Lemma 2.8 to matrix $\mathcal{M}^{r+s}$ we obtain that

$$\mathcal{M}^{r+s} = \begin{bmatrix} M_{r+s+1} & -2M_{r+s} \\ M_{r+s} & -2M_{r+s-1} \end{bmatrix}.$$

Alternatively, the properties of matrix multiplication indicate that $\mathcal{M}^{r+s} = \mathcal{M}^r \cdot \mathcal{M}^s$, which can be expressed as follows:

$$\begin{aligned} \mathcal{M}^r \cdot \mathcal{M}^s &= \begin{bmatrix} M_{r+1} & -2M_r \\ M_r & -2M_{r-1} \end{bmatrix} \begin{bmatrix} M_{s+1} & -2M_s \\ M_s & -2M_{s-1} \end{bmatrix} \\ &= \begin{bmatrix} M_{r+1}M_{s+1} - 2M_rM_s & -2M_{r+1}M_s - 2M_rM_{s-1} \\ M_rM_{s+1} - 2M_{r-1}M_s & -2M_rM_s + 2^2M_{r-1}M_{s-1} \end{bmatrix}. \end{aligned}$$

The result is a consequence of the equality between matrices. □

3 Tricomplex Mersenne: the Binet formula, vector second order recurrence and generating function

In this section, we define the tricomplex Mersenne numbers and present the vector second order recurrence. In addition, the vector Binet formulas are provided, and several identities are established, along with the ordinary generating and exponential generating functions of the tricomplex Mersenne sequence.

According to Definition 1.1, remember that in the ring $\mathbb{T}$, we present the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$ represented by:

$$TM_n^* = (M_n^*, M_{n+1}^*, M_{n+2}^*) ,$$

where $\{M_n^*\}_{n\geq 1}$ is the Mersenne-type sequence for all natural numbers n.

A direct calculation shows that:

Proposition 3.1. *The tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$ satisfies the recurrence relation*

$$TM_{n+2}^* = 3TM_{n+1}^* - 2TM_n^*;\ n \geq 0 , \tag{6}$$

with the initial conditions $TM_0^* = (M_0^*, M_1^*, M_2^*)$ and $TM_1^* = (M_1^*, M_2^*, M_3^*)$, and where $\{M_n^*\}_{n\geq 1}$ is the Mersenne-type sequence.

Therefore, the vector sequence $\{TM_n^*\}_{n\geq 0}$, which is derived from the Mersenne-type sequence $\{M_n^*\}_{n\geq 0}$, is also a kind of Mersenne sequence because it follows a "similar" recurrence relation to that of the Mersenne-type sequence.

Let $\mathcal{GF}_{TM_n^*}(x) = \left(GF_{M_n^*}(x), GF_{M_{n+1}^*}(x), GF_{M_{n+2}^*}(x)\right)$ be the ordinary generating function for the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$.

Next, we will present the generating function for the tricomplex Mersenne sequence.

Proposition 3.2. *The ordinary generating function for the tricomplex Mersenne $\{TM_n^*\}_{n\geq 0}$, denoted by $\mathcal{GF}_{TM_n^*}(x)$, is given by*

$$\left(\frac{M_0^*+(-1)^{\xi+1}M_1^*x}{2x^2-3x+1}, \quad \frac{3M_0^*+(-1)^{\xi+1}M_1^*-2M_0^*x}{2x^3-3x^2+x}, \quad \frac{3M_0^*+[(-1)^{\xi+1}-1]M_1^*+(3M_1^*-2M_0^*)x-2M_1^*x^2}{2x^3-3x^2+x}\right) , \tag{7}$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$, and where $\{M_n^*\}_{n\geq 1}$ is the Mersenne-type sequence.

Proof. The first coordinate of the vector Equation (7) is a direct application of Lemma 2.2.

Note that $GF_{M_{n+1}^*}(x) = \sum_{n=0}^{\infty} M_{n+1}^* x^n$. Then, by expanding equation $xGF_{M_{n+1}^*}(x)$, we obtain

$$\begin{aligned} xGF_{M_{n+1}^*}(x) &= M_1^*x + M_2^*x^2 + M_3^*x^3 + \dots \\ &= GF_{M_n}(x) - M_0^* \\ &= \frac{[3M_0^* + (-1)^{\xi+1}M_1^*]x - 2M_0^*x^2}{2x^2 - 3x + 1} , \end{aligned}$$

this proves the second coordinate. The proof of the third coordinate is done similarly. □

Let $\mathcal{E}_{TM_n^*}(x) = \left(E_{M_n^*}(x), E_{M_{n+1}^*}(x), E_{M_{n+2}^*}(x)\right)$ be the exponential generating function for the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$, where $\{M_n^*\}_{n\geq 1}$ is the Mersenne-type sequence.

Combining the Definition 1.1 and Lemma 2.3, we obtain the next result.

Proposition 3.3. *For all $n \geq 0$ the exponential generating function for the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$ is*

$$\mathcal{E}_{TM_n^*}(x) = [e^{2x} + (-1)^{\xi} e^x] \left(1, 2, 2^2\right) , \tag{8}$$

where $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. The first coordinate of the vector in Equation (8) directly applies Lemma 2.3.

$$GE_{M_n^*}(x) = \sum_{n=0}^{\infty} \frac{M_n x^n}{n!} = e^{2x} + (-1)^{\xi} e^x .$$

Note the following $GE_{M_{n+1}^*}(x) = \sum_{n=0}^{\infty} \frac{M_{n+1}^* x^n}{n!}$. Then, by Equation (4), we obtain

$$\begin{aligned} GE_{M_{n+1}^*}(x) &= \sum_{n=0}^{\infty} \frac{(2^{n+1} + (-1)^{\xi}) x^n}{n!} \\ &= 2e^{2x} + (-1)^{\xi} e^x , \end{aligned}$$

this proves the second coordinate of the vector in Equation (8). The proof of the third coordinate is done similarly. □

4 Elementary properties for tricomplex Mersenne numbers

In this section, we present and establish several classical identities associated with the tricomplex Mersenne sequence, denoted by $\{TM_n^*\}_{n\geq 0}$. This includes, for instance, the Tagiuri-Vajda identity, along with various derived forms and extensions. These identities serve as foundational tools for exploring the properties and applications of the tricomplex Mersenne sequence in greater depth.

We start the convolution identity for the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$.

Theorem 4.1. *Let $\{TM_n^*\}_{n\geq 0}$ be the tricomplex Mersenne sequence, and n, k be any non-negative integers. We have the following identity:*

$$\begin{aligned} & TM_k^* \times TM_{n-1}^* + TM_{k+1}^* \times TM_n^* \\ = & \left(5(M_{k+n-1} + 2M_{k+n+2}), 5(M_{k+n} + 2M_{k+n+3}), 15M_{k+n+1}\right) + A(1,1,1) . \end{aligned}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence, $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$, and $A = (-1)^{\xi} 3\left[M_k + M_{k+1} + M_{k+2} + M_{n-1} + M_n + M_{n+1} + 6 + 3(-1)^{\xi}\right] + 12$.

Proof. Note that

$$\begin{aligned} & TM_k^* \times TM_{n-1}^* + TM_{k+1}^* \times TM_n^* \\ = & (M_k^*, M_{k+1}^*, M_{k+2}^*) \times (M_{n-1}^*, M_n^*, M_{n+1}^*) \\ & + (M_{k+1}^*, M_{k+2}^*, M_{k+3}^*) \times (M_n^*, M_{n+1}^*, M_{n+2}^*) . \end{aligned} \tag{9}$$

The first component of the vector defined in Equation (9) is

$$\begin{aligned} & (M_k^* M_{n-1}^* + M_{k+1}^* M_{n+1}^* + M_{k+2}^* M_n^*) + (M_{k+1}^* M_n^* + M_{k+2}^* M_{n+2}^* + M_{k+3}^* M_{n+1}^*) \\ = \; & (M_k^* M_{n-1}^* + M_{k+1}^* M_n^*) + (M_{k+1}^* M_{n+1}^* + M_{k+2}^* M_{n+2}^*) + (M_{k+2}^* M_n^* + M_{k+3}^* M_{n+1}^*) . \end{aligned}$$

By Proposition 2.4, we have

$$\begin{aligned} & (M_k^* M_{n-1}^* + M_{k+1}^* M_n^*) + (M_{k+1}^* M_{n+1}^* + M_{k+2}^* M_{n+2}^*) + (M_{k+2}^* M_n^* + M_{k+3}^* M_{n+1}^*) \\ = \; & [5M_{k+n-1} + (-1)^\xi 3\big(M_k + M_{n-1} + 2 + (-1)^\xi\big) + 4] \\ & + [5M_{k+n+2} + (-1)^\xi 3\big(M_{k+1} + M_{n+1} + 2 + (-1)^\xi\big) + 4] \\ & + [5M_{k+n+2} + (-1)^\xi 3\big(M_{k+2} + M_n + 2 + (-1)^\xi\big) + 4] \\ = \; & 5(M_{k+n-1} + 2M_{k+n+2}) + 12 \\ & + (-1)^\xi 3\big[M_k + M_{k+1} + M_{k+2} + M_{n-1} + M_n + M_{n+1} + 6 + 3(-1)^\xi\big] , \end{aligned}$$

This proves the first component of the vector defined in Equation (9).

Making use of Proposition 2.4, the second component of the vector defined in Equation (9) is

$$\begin{aligned} & (M_{k+2}^* M_{n+1}^* + M_k^* M_n^* + M_{k+1}^* M_{n-1}^*) + (M_{k+3}^* M_{n+2}^* + M_{k+1}^* M_{n+1}^* + M_{k+2}^* M_n^*) \\ = \; & (M_{k+2}^* M_{n+1}^* + M_{k+3}^* M_{n+2}^*) + (M_k^* M_n^* + M_{k+1}^* M_{n+1}^*) + (M_{k+1}^* M_{n-1}^* + M_{k+2}^* M_n^*) \\ = \; & [5M_{k+n+3} + (-1)^\xi 3\big(M_{k+2} + M_{n+1} + 2 + (-1)^\xi\big) + 4] \\ & + [5M_{k+n} + (-1)^\xi 3\big(M_k + M_n + 2 + (-1)^\xi\big) + 4] \\ & + [5M_{k+n} + (-1)^\xi 3\big(M_{k+1} + M_{n-1} + 2 + (-1)^\xi\big) + 4] \\ = \; & 5(M_{k+n+3} + 2M_{k+n}) + 12 \\ & + (-1)^\xi 3\big[M_k + M_{k+1} + M_{k+2} + M_{n-1} + M_n + M_{n+1} + 6 + 3(-1)^\xi\big] , \end{aligned}$$

This proves the second component of the vector defined in Equation (9).

In a similar way, the third component of the vector defined in Equation (9) is

$$\begin{aligned} & (M_{k+1}^* M_n^* + M_{k+2}^* M_{n+1}^*) + (M_k^* M_{n+1}^* + M_{k+1}^* M_{n+2}^*) + (M_{k+2}^* M_{n-1}^* + M_{k+3}^* M_n^*) \\ = \; & 15M_{k+n+1} + 12 \\ & + (-1)^\xi 3\big[M_k + M_{k+1} + M_{k+2} + M_{n-1} + M_n + M_{n+1} + 6 + 3(-1)^\xi\big] . \end{aligned}$$

Make $A = (-1)^\xi 3\big[M_k + M_{k+1} + M_{k+2} + M_{n-1} + M_n + M_{n+1} + 6 + 3(-1)^\xi\big] + 12$; this completes the proof. □

Now, the Tagiuri-Vajda Identity for the tricomplex Mersenne sequence $\{TM_n^*\}_{n\geq 0}$.

Theorem 4.2. *Let $\{TM_n^*\}_{n\geq 0}$ be the tricomplex Mersenne sequence, and n, s, k be any non-negative integers. We have the following identity:*

$$TM_{n+s}^* \times TM_{n+k}^* - TM_n^* \times TM_{n+s+k}^* = (-1)^{\xi+1} 7 \cdot 2^n M_s M_k \cdot (1,\ 1,\ 1) , \tag{10}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. Note that

$$\begin{aligned} & TM_{n+s}^* \times TM_{n+k}^* - TM_n^* \times TM_{n+s+k}^* \\ = \; & (M_{n+s}^*, M_{n+s+1}^*, M_{n+s+2}^*) \times (M_{n+k}^*, M_{n+k+1}^*, M_{n+k+2}^*) \\ = \; & (\ M_{n+s}^* \cdot M_{n+k}^* + M_{n+s+1}^* \cdot M_{n+k+2}^* + M_{n+s+2}^* \cdot M_{n+k+1}^* , \\ & M_{n+s+2}^* \cdot M_{n+k+2}^* + M_{n+s}^* \cdot M_{n+k+1}^* + M_{n+s+1}^* \cdot M_{n+k}^* , \\ & M_{n+s+1}^* \cdot M_{n+k+1}^* + M_{n+s}^* \cdot M_{n+k+2}^* + M_{n+s+2}^* \cdot M_{n+k}^* \ ) , \end{aligned} \tag{11}$$

and,

$$\begin{aligned} & TM_n^* \times TM_{n+s+k}^* \\ = \; & (M_n^*, M_{n+1}^*, M_{n+2}^*) \times (M_{n+s+k}^*, M_{n+s+k+1}^*, M_{n+s+k+2}^*) \\ = \; & (\, M_n^* \cdot M_{n+s+k}^* + M_{n+1}^* \cdot M_{n+s+k+2}^* + M_{n+2}^* \cdot M_{n+s+k+1}^* \,, \\ & M_{n+2}^* \cdot M_{n+s+k+2}^* + M_n^* \cdot M_{n+s+k+1}^* + M_{n+1}^* \cdot M_{n+s+k}^* \,, \\ & M_{n+1}^* \cdot M_{n+s+k+1}^* + M_n^* \cdot M_{n+s+k+2}^* + M_{n+2}^* \cdot M_{n+s+k}^* \,) \,. \end{aligned} \tag{12}$$

To obtain the first coordinate of the vector defined in Equation (10), we subtract Equation (12) from Equation (11) as follows:

$$\begin{aligned} & (M_{n+s}^* \cdot M_{n+k}^* - M_n^* \cdot M_{n+s+k}^*) + (M_{n+s+1}^* \cdot M_{n+k+2}^* - M_{n+1}^* \cdot M_{n+s+k+2}^*) \\ & +(M_{n+s+2}^* \cdot M_{n+k+1}^* - M_{n+2}^* \cdot M_{n+s+k+1}^*) \\ = \; & \big(M_{n+s}^* \cdot M_{n+k}^* - M_n^* \cdot M_{n+s+k}^*\big) \\ & +\big(M_{(r+1)+s}^* \cdot M_{(r+1)+(k+1)}^* - M_{(r+1)}^* \cdot M_{(r+1)+s+(k+1)}^*\big) \\ & +\big(M_{(r+2)+s}^* \cdot M_{(r+2)+(k-1)}^* - M_{(r+2)}^* \cdot M_{(r+2)+s+(k-1)}^*\big). \end{aligned}$$

By Proposition 2.5, we have that

$$\begin{aligned} & (M_{n+s}^* \cdot M_{n+k}^* - M_n^* \cdot M_{n+s+k}^*) + (M_{n+s+1}^* \cdot M_{n+k+2}^* - M_{n+1}^* \cdot M_{n+s+k+2}^*) \\ & +(M_{n+s+2}^* \cdot M_{n+k+1}^* - M_{n+2}^* \cdot M_{n+s+k+1}^*) \\ = \; & \big((-1)^{\xi} 2^r M_s M_k\big) + \big((-1)^{\xi} 2^{r+1} M_s M_{k+1}\big) + \big((-1)^{\xi} 2^{r+2} M_s M_{k-1}\big) \\ = \; & (-1)^{\xi+1} 7 \cdot 2^n M_s M_k \end{aligned}$$

This completes the proof for the first coordinate of the vector defined in Equation (10). The proofs for the second and third coordinates follow similarly by applying Proposition 2.5, which uses the Tagiuri-Vajda identity for the Mersenne-type sequence. □

The other identities will follow as a consequence of Theorem 4.2, as we will demonstrate below.

Proposition 4.3. *(d'Ocagne's Identity) Let $\{TM_n^*\}_{n\geq 0}$ be tricomplex Mersenne sequence, and m, n be any natural numbers. For $m \geq n$ we have*

$$TM_m^* \times TM_{n+1}^* - TM_{m+1}^* \times TM_n^* = (-1)^{\xi} 7 \cdot 2^n M_{m-n} \cdot (1,1,1) \,,$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. Let $s = m - n$ and $k = 1$ in Theorem 4.2, then

$$(-1)^{\xi} 7 \cdot 2^n M_{m-n} M_1.$$

□

Similar to Proposition 4.3 we have the Catalan Identity.

Proposition 4.4. *Let m, n be any natural numbers and $\{TM_n^*\}_{n\geq 0}$ be tricomplex Mersenne sequence. For $m \geq n$ we have*

$$TM_{m-n}^* \times TM_{m+n}^* - (TM_n^*)^2 = \; (-1)^{\xi+1} 7 \cdot (M_n)^2 \cdot (1,\; 1,\; 1) \,,$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. Just take $s = n$ and $k = -n$ in Theorem 4.2 we have that

$$(-1)^{\xi}7 \cdot 2^n M_n M_{-n} = (-1)^{\xi+1}7 \cdot (M_n)^2 .$$

Since $M_{-n} = \frac{-M_n}{2^n}$, the result follows. □

Making $n = 1$ and as $M_1 = 1$, it follows directly from Proposition 4.4, without much effort, that:

Corollary 4.5. *(Cassini-Sinsom's Identity) For all $m \geq 1$, we have*

$$(TM_n^*)^2 - TM_{m-1}^* \times TM_{m+1}^* = (-1)^{\xi}7 \cdot (1,\ 1,\ 1) ,$$

where $\xi(M^*) = 0$ if $\{M_n\}_{n\geq 0} = \{M_n\}_{n\geq 0}$ and $\xi(M^*) = 1$ if $\{M_n\}_{n\geq 0} = \{m_n\}_{n\geq 0}$,

In conclusion, this section presents a specific result for the tricomplex Mersenne sequence, restricted to the Mersenne sequence, derived from the Proposition 2.9.

Proposition 4.6. *For any natural numbers r and s, the tricomplex Mersenne sequence $\{TM_n\}_{n\geq 0}$ satisfies*

$$TM_r \times TM_{s+1} - 2TM_{r-1} \times TM_s = \big(M_{r+s} + 2M_{r+s+3},\ M_{r+s+4} + 2M_{r+s+1},\ 3M_{r+s+2}\big), \tag{13}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. See that

$$\begin{aligned} TM_r \times TM_{s+1} =& \big(M_r,\ M_{r+1},\ M_{r+2}\big)\times\big(M_{s+1},\ M_{s+2},\ M_{s+3}\big) \\ =& M_{r+1}M_{s+2} + M_r M_{r+3} + M_{r+2}M_{s+1}\big) , \end{aligned} \tag{14}$$

and,

$$\begin{aligned} 2TM_{r-1} \times TM_s =& 2\big(M_{r-1}M_s + M_r M_{s+2} + M_{r+1}M_{s+1}, \\ =& M_r M_{s+1} + M_{r-1}M_{s+2} + M_{r+1}M_s\big). \end{aligned} \tag{15}$$

To obtain the first coordinate of Equation (13), we subtract Equation (14) from Equation (15); that is,

$$\begin{aligned} &(M_r M_{n+1} - 2M_{r-1}M_n) + (M_{r+1}M_{n+3} - 2M_r M_{n+2}) \\ &+ (M_{r+2}M_{rn+2} - 2M_{r+1}M_{n+1}) \\ =&(M_r M_{n+1} - 2M_{r-1}M_n) + (M_{(m+1)}M_{(n+1)+2} - 2M_r M_{(n+2)}) \\ &+ (M_{(m+2)}M_{(n+1)+1} - 2M_{(m-1)+2}M_{n+1}) . \end{aligned}$$

According to Proposition 2.9, we have

$$\begin{aligned} &(M_r M_{n+1} - 2M_{r-1}M_n) + (M_{r+1}M_{n+3} - 2M_r M_{n+2}) \\ &+ (M_{r+2}M_{rn+2} - 2M_{r+1}M_{n+1}) \\ =&M_{r+n} + M_{(m+1)+(n+2)} + M_{(m+2)+(n+1)} \\ =&M_{r+n} + 2M_{r+n+3} \end{aligned}$$

This proves the first coordinate. Proofs of the second and third coordinates are done similarly using Proposition 2.9. □

5 Sum of terms involving the tricomplex Mersenne numbers

In this section, we present results on the partial sums of terms of the tricomplex Mersenne numbers with n natural number. Now, we consider the sequence of partial sums $\sum_{k=0}^{n} TM_k^* = TM_0^* + TM_1^* + TM_2^* + \cdots + TM_n^*$, for $n \geq 0$, where $\{TM_n^*\}_{n\geq 0}$ is the tricomplex Mersenne sequence.

In the next result we establish the partial sum of tricomplex Mersenne sequence.

Proposition 5.1. *Let $\{TM_n^*\}_{n\geq 0}$ be the tricomplex Mersenne sequence, we have the following formula:*

$$\begin{aligned}\sum_{k=0}^{n} TM_k^* &= (2^{n+1} + (-1)^{\xi} n,\ 2^{n+2} + (-1)^{\xi}(n+1),\\ &\qquad 2^{n+3} + (-1)^{\xi}(n+2)) - (0, M_0^*, M_0^* + M_1^*),\end{aligned}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. We have that

$$\begin{aligned}\sum_{k=0}^{n} TM_k^* &= \sum_{k=0}^{n} \left(M_k^*, M_{k+1}^*, M_{k+2}^*\right)\\ &= \left(\sum_{k=0}^{n} M_k^*, \sum_{k=1}^{n+1} M_k^*, \sum_{k=2}^{n+2} M_k^*\right).\end{aligned}$$

In the first component, we have, according to Lemma 2.6,

$$\sum_{k=0}^{n} M_k^* = 2^{n+1} - 1 + (n+1)(-1)^{\xi}.$$

In the second component, using Lemma 2.6, we have:

$$\begin{aligned}\sum_{k=1}^{n+1} M_k^* &= \sum_{k=0}^{n+1} M_k^* - M_0^*\\ &= 2^{n+2} - 1 + (n+2)(-1)^{\xi} - M_0^*.\end{aligned}$$

In a similar way, in the second component, we have:

$$\sum_{k=2}^{n+2} M_k^* = 2^{n+3} - 1 + (n+3)(-1)^{\xi} - M_0^* - M_1^*,$$

and we get the result. □

Now, we present the partial sums of the tricomplex Mersenne sequence for odd indices and even indices separately.

Proposition 5.2. *Let $\{TM_n^*\}_{n\geq 0}$ be the tricomplex Mersenne sequence, we have the following formulas:*

$$\begin{aligned}
(a)\ \sum_{k=0}^{n} TM_{2k}^* &= \left(\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1),\right.\\
&\quad \left. 2\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1), \frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_0^*\right),\\
(b)\ \sum_{k=0}^{n} TM_{2k+1}^* &= \left(2\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1),\right.\\
&\quad \left. \frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_0^*, 2\frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_1^*\right),
\end{aligned}$$

where $\{M_n^*\}_{n\geq 0}$ is the Mersenne-type sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. (a) Combining the definition of tricomplex Mersenne numbers and partial sum, we have

$$\begin{aligned}
\sum_{k=0}^{n} TM_{2k}^* &= TM_0^* + TM_2^* + \ldots + TM_{2n}^*\\
&= (M_0^*, M_1^*, M_2^*) + (M_2^*, M_3^*, M_4^*) + \ldots + (M_{2n}^*, M_{2n+1}^*, M_{2(n+1)}^*)\\
&= \left(\sum_{k=0}^{n} M_{2k}^*, \sum_{k=0}^{n} M_{2k+1}^*, \sum_{k=1}^{n+1} M_{2k}^*\right)
\end{aligned}$$

Make use of Proposition 2.7, items (a) and (b), we have

$$\begin{aligned}
\sum_{k=0}^{n} TM_{2k}^* &= \left(\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1),\right.\\
&\quad \left. 2\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1), \frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_0^*\right),
\end{aligned}$$

and we get the result.

(b) Similarly,

$$\begin{aligned}
\sum_{k=1}^{n} TM_{2k+1}^* &= TM_1^* + TM_3^* + \ldots + TM_{2n+1}^*\\
&= \left(\sum_{k=0}^{n} M_{2k+1}^*, \sum_{k=1}^{n+1} M_{2k}^*, \sum_{k=1}^{n+1} M_{2k+1}^*\right)\\
&= \left(2\frac{4^{n+1}-1}{3}+(-1)^{\xi}(n+1), \frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_0^*,\right.\\
&\quad \left. 2\frac{4^{n+2}-1}{3}+(-1)^{\xi}(n+2)-M_1^*\right)
\end{aligned}$$

which verifies the result. □

The alternating sum of the first n terms of a sequence is defined as:

$$\sum_{k=0}^{n}(-1)^k TM_k^* = TM_0^* - TM_1^* + TM_2^* - TM_3^* + \cdots + (-1)^n TM_n^*,$$

where $\{TM_n^*\}_{n\geq 0}$ is the tricomplex Mersenne sequence.

An immediate consequence of the previous results is the theorem presented below. This result naturally emerges from the established relationships and provides additional support for the conclusions drawn from Proposition 5.2.

Proposition 5.3. *For $m \geq 0$, we have the following formulas:*

$$(a) \quad \sum_{k=0}^{m}(-1)^k TM_k^* = \left(-\tfrac{4^{n+1}-1}{3}, -2\tfrac{4^{n+1}}{3} - (-1)^{\xi} + M_0^*, -\tfrac{4^{n+2}-1}{3} - M_0^* + M_1^*\right);$$

$$\text{if } m = 2n+1 \text{ is odd, and}$$

$$(b) \quad \sum_{k=0}^{m}(-1)^k TM_k^* = \left(2\tfrac{4^{n+1}}{3} + (-1)^{\xi}, \tfrac{4^{n+2}-1}{3} + M_0^*, 2\tfrac{4^{n+2}}{3} + (-1)^{\xi} - M_0^* + M_1^*\right);$$

$$\text{if } m = 2n \text{ is even,}$$

where $\{TM_n^*\}_{n\geq 0}$ is the tricomplex Mersenne sequence, $\{M_n^*\}_{n\geq 0}$ is the Mersenne-type sequence, and $\xi = 1$ if $\{M_n^*\}_{n\geq 0} = \{M_n\}_{n\geq 0}$, and $\xi = 0$ if $\{M_n^*\}_{n\geq 0} = \{m_n\}_{n\geq 0}$.

Proof. (a) First, suppose that m is an odd natural number, that is $m = 2n+1$ for some natural number m. Thus

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k TM_k^* &= TM_0^* - TM_1^* + TM_2^* - TM_3^* + \cdots + TM_{2n}^* - TM_{2n+1}^* \\ &= (TM_0^* + TM_2^* + \cdots + TM_{2n}^*) - (TM_1^* + TM_3^* + \cdots + TM_{2n+1}^*) \\ &= \sum_{k=0}^{n} TM_{2k}^* - \sum_{k=0}^{n} TM_{(2k+1)}^*\end{aligned}$$

We obtain the result using Proposition 5.2, items (b) and (c), it follows that:

$$\begin{aligned}&\sum_{k=0}^{2m+1}(-1)^k TM_k^* \\ = &\left(-\frac{4^{n+1}-1}{3}, -2\frac{4^{n+1}}{3} - (-1)^{\xi} + M_0^*, -\frac{4^{n+2}-1}{3} - M_0^* + M_1^*\right),\end{aligned}$$

as required.

(b) In this case, we consider $n = 2m$, an even natural number. So

$$\begin{aligned}\sum_{k=0}^{2(n+1)}(-1)^k TM_k^* &= TM_0^* - TM_1^* + TM_2^* - TM_3^* + \cdots + TM_{2n}^* - TM_{2n+1}^* + TM_{2n+2}^* \\ &= \sum_{k=0}^{n+1} TM_{2k}^* - \sum_{k=0}^{n} TM_{(2k+1)}^*\end{aligned}$$

As in item (a), apply Proposition 5.2. □

6 Final Considerations

In this work, we introduce a new family of arithmetic sequences connected to Mersenne numbers, focusing specifically on the study of a Mersenne-type sequence within the tricomplex ring $\mathbb{T}$. Through this exploration, we expand the understanding of tricomplex numbers by developing

and analyzing innovative mathematical concepts and results. The findings presented here reflect significant progress in advancing the theory and applications of tricomplex numbers within the study. It is evident that modifying the initial conditions, such as the recurrence or the initial values, will result in the emergence of novel sequences that warrant further investigation. We encourage interested readers to delve deeper into these extensions. Additionally, our future research aims to generalize the Mersenne-type sequence in this ring, thereby expanding the frontiers of mathematical knowledge and potential applications.

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Declarations

Conflict of interest

The authors declare that they have no conflict of interest.

Authors' contributions

All authors contributed equally to this work.

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