Latin American Journal of Mathematics
Vol. 5, No. 1, 2026, pp. 39–50
10.14244/lajm.v5i1.94
ISSN: 2965-0798
www.lajm.ufscar.br

On ρ-statistical convergence in gradual normed linear spaces

Shyamal Debnath [1], **Santonu Debnath** [2], **and Chiranjib Choudhury** [1]

[1] *Department of Mathematics, Tripura University (A Central University), INDIA*
[2] *Department of Mathematics, Dhamma Dipa International Buddhist University, INDIA*

E-mails: shyamalnitamath@gmail.com, santonudebnath16@gmail.com, chiranjibchoudhury123@gmail.com

Received: 12 March 2026 • **Accepted:** 2 July 2026 • **Published:** 3 August 2026

Abstract

In this article, we present the notion of ρ-statistical convergence in the context of gradual normed linear spaces and explore connections between statistical boundedness in gradual normed linear spaces and ρ-statistical boundedness. Additionally, we introduce the concept of ρ-statistical Cauchy sequences in gradual normed spaces and establish the correlation between gradual ρ-statistical convergence and gradual ρ-statistical Cauchy sequences.

Keywords: Statistical convergence, ρ-statistical convergence, ρ-statistical boundedness, gradual normed linear spaces.

MSC2020: 03E72, 40A35, 40A05.

1 Introduction, Definitions and Preliminaries

In 1965, Zadeh [28] introduced the pioneering concept of fuzzy sets, expanding upon classical set theory. Today, its widespread applications span various scientific and engineering domains. The term "fuzzy number" assumes a crucial role within fuzzy set theory, representing a generalization of intervals rather than precise numerical values. Notably, fuzzy numbers diverge from classical numbers in their adherence to algebraic properties, prompting debate among researchers regarding their classification. Consequently, some scholars prefer the term "fuzzy intervals" to delineate this distinction. In an effort to mitigate this ambiguity, Fortin et al. [17] proposed the notion of gradual real numbers as constituents of fuzzy intervals. Gradual real numbers are distinguished by their assignment functions, typically defined within the interval $(0, 1]$. Intriguingly, every real number can be construed as a gradual number with a constant assignment function. Importantly, gradual real numbers align with the algebraic properties of classical real numbers and find utility in computational and optimization contexts.

In 2011, Sadeqi and Azari [24] first introduced the concept of gradual normed linear space. Their exploration encompassed diverse properties of this space, scrutinizing it from both algebraic and topological standpoints. Subsequent advancements in this area have been propelled by the works of Ettefagh et al. [14, 15], Choudhury and Debnath [7, 8, 9], and many others. For an extensive study on gradual real numbers, one may refer to [10, 21], which serves as a comprehensive resource featuring additional references for further exploration.

The opinion of statistical convergence depends on the density of subsets of the natural number $\mathbb{N}$. We say that $\delta(E)$ is the density of a subset E of $\mathbb{N}$ if the following limit exists:

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$. Then, the sequence $s = (s_k)$ is statistically convergent [18] to ℓ if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |s_k - \ell| \geq \varepsilon\}) = 0.$$

for each $\varepsilon > 0$. For more study on statistical convergence, one may refer to [11, 20, 22, 23, 26, 25, 27]. The ρ-density [6] of a set $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{\rho_n} |\{ k \leq n : k \in K\}|, \ \limsup_n \frac{\rho_n}{n} < \infty, \Delta\rho_n = O(1) \text{ and } \Delta\rho_n = \rho_{n+1} - \rho_n \quad (1.1)$$

provided this limit exists, where, $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that for each positive integer n.

If (s_k) is a sequence such that s_k holds property $P(k)$ for all k except a set of ρ-density zero, then we say that s_k holds $P(k)$ for "almost all k according to ρ" and we denote this by "$a.a.k\ (\rho)$".

A sequence (s_k) is said to be ρ-statistically convergent [6] to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |s_k - \ell| \geq \varepsilon\}| = 0$$

where, $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ satisfying condition (1.1). For more study on ρ-statistical convergence, one may refer to [4, 5]. If $\rho_n = n$ for all $n \in \mathbb{N}$, then ρ-statistical convergence coincides with statistical convergence. The set of all ρ-statistically convergent sequences will be denoted by S_ρ. In pure mathematics, ρ-statistical convergence serves as a fundamental tool in sequence space theory and summability theory. It provides a flexible mechanism to generalize classical limits and analyze structural properties such as inclusion relations, dual spaces and matrix transformations between various sequence spaces [16, 1]. Within fuzzy and gradual functional analysis, this convergence method is essential for investigating the behavior of bounded linear operators. It significantly extends the classical reach of Fuzzy Korovkin-type approximation theorems were established by Baxhaku et al. [29] in 2022., allowing researchers to establish uniform approximation properties for sequences of positive linear operators where traditional analytical convergence criteria fail to hold Furthermore, it plays a vital role in topological vector spaces and fixed point theory within non classical normed structures. By utilizing the scaling (ρ_n), mathematicians establish more refined topological densities to study contractive mappings fixed point invariants and structural completeness across fuzzy, intuitionistic fuzzy and gradual normed linear spaces [24].

The real number sequence $s = (s_k)$ is statistically bounded [19] if there exists a number $M \geq 0$ such that

$$\delta(\{k : |s_k| > M\}) = 0.$$

The set of all statistically bounded sequences will be denoted by $S(b)$.

It can be shown that every bounded sequence is statistically bounded, but the converse is not true. For this consider a sequence $s = (s_k)$ defined by

$$s_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 1, & \text{if } k \text{ is not a square.} \end{cases}$$

Clearly, (s_k) is not a bounded sequence, but it is statistically bounded. Bhardwaj et al. [2, 3] and Et et al. [12, 13] generalized the concept of statistical boundedness.

Definition 1.1. *[17] A gradual real number $\tilde{r}$ is defined by an assignment function $\mathscr{A}_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number is said to be non-negative if for every $\varphi \in (0,1]$, $\mathscr{A}_{\tilde{r}}(\varphi) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.*

In [17], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

Definition 1.2. *Let $*$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $\mathscr{A}_{\tilde{r}_1}$ and $\mathscr{A}_{\tilde{r}_2}$ respectively. Then $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $\mathscr{A}_{\tilde{r}_1 * \tilde{r}_2}$ given by $\mathscr{A}_{\tilde{r}_1 * \tilde{r}_2}(\varphi) = \mathscr{A}_{\tilde{r}_1}(\varphi) * \mathscr{A}_{\tilde{r}_2}(\varphi)\ \forall \varphi \in (0,1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $\lambda \tilde{r} (\lambda \in \mathbb{R})$ are defined by*

$$\mathscr{A}_{\tilde{r}_1 + \tilde{r}_2}(\varphi) = \mathscr{A}_{\tilde{r}_1}(\varphi) + \mathscr{A}_{\tilde{r}_2}(\varphi) \quad and \quad \mathscr{A}_{\lambda \tilde{r}}(\varphi) = \lambda \mathscr{A}_{\tilde{r}}(\varphi)\ \ \forall \varphi \in (0,1].$$

For any real number $p \in \mathbb{R}$, the constant gradual real number $\tilde{p}$ is defined by the constant assignment function $\mathscr{A}_{\tilde{p}}(\varphi) = p$ for any $\varphi \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $\mathscr{A}_{\tilde{0}}(\varphi) = 0$ and $\mathscr{A}_{\tilde{1}}(\varphi) = 1$ respectively. One can easily verify that $G(\mathbb{R})$ with gradual addition and multiplication forms a real vector space [17].

Definition 1.3. *[24] Let X be a real vector space. The function $||\cdot||_{\mathscr{G}} : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X if for every $\varphi \in (0,1]$, the following three conditions are true for any $s, t \in X$:*
(G_1) $\mathscr{A}_{||s||_{\mathscr{G}}}(\varphi) = \mathscr{A}_{\tilde{0}}(\varphi)$ iff $s = 0$;
(G_2) $\mathscr{A}_{||\lambda s||_{\mathscr{G}}}(\varphi) = |\lambda| \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi)$ for any $\lambda \in \mathbb{R}$;
(G_3) $\mathscr{A}_{||s+t||_{\mathscr{G}}}(\varphi) \leq \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi) + \mathscr{A}_{||t||_{\mathscr{G}}}(\varphi)$.

The pair $(X, ||\cdot||_{\mathscr{G}})$ is called a gradual normed linear space (GNLS).

Example 1.4. *[24] Let $X = \mathbb{R}^m$ and for $s = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m, \varphi \in (0,1]$, define $||\cdot||_{\mathscr{G}}$ by $\mathscr{A}_{||s||_{\mathscr{G}}}(\varphi) = e^{\varphi} \sum_{i=1}^{m} |s_i|$. Then $||\cdot||_{\mathscr{G}}$ is a gradual norm on $\mathbb{R}^m$ and $(\mathbb{R}^m, ||\cdot||_{\mathscr{G}})$ is a gradual normed linear space.*

Definition 1.5. *[24] Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be gradually convergent to $s \in X$ if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, there exists $N(\varphi) \in \mathbb{N}$ such that $\mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) < \varepsilon,\ \forall k \geq N(\varphi)$.*

Definition 1.6. *[15] Let $(X, ||\cdot||_{\mathscr{G}})$ be a GNLS. Then a sequence (s_k) in X is said to be gradually bounded if for every $\varphi \in (0,1]$, there exists $M = M(\varphi) > 0$, such that $\mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) < M,\ \forall k \in \mathbb{N}$.*

Definition 1.7. *[24] Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be gradual Cauchy if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, there exists $N(\varphi) \in \mathbb{N}$ such that $\mathscr{A}_{||s_k - s_j||_{\mathscr{G}}}(\varphi) < \varepsilon\ \forall k, j \geq N(\varphi)$.*

Definition 1.8. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be a gradual statistical convergent to $s \in X$ if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$ has natural density zero. Symbolically, $s_k \xrightarrow{st-||\cdot||_{\mathscr{G}}} s$.*

2 Main Results

In this section, we will present the main results of the paper. We begin with the following definition:

Definition 2.1. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then, (s_k) is said to be gradual ρ-statistical convergent to $s \in X$ if for every $\varphi \in (0,1]$ and $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$ has ρ-density zero. Symbolically we write, $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$.*

Theorem 2.2. *Let $(X, ||\cdot||_{\mathscr{G}})$ be a GNLS. If a sequence (s_k) is gradually convergent to $s \in X$, then (s_k) is gradual ρ-statistical convergent to $s \in X$.*

Proof. Since (s_k) is gradually convergent to s, so the set $\{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$ is a finite set and so has ρ-density zero. □

But the converse of Theorem 2.2 is not true. Example 2.3 clarify this fact.

Example 2.3. *Let $X = \mathbb{R}^m$ and $||\cdot||_{\mathscr{G}}$ be the norm defined in Example 1.4. Consider the sequence (s_k) in $\mathbb{R}^m$ defined as*

$$s_k = \begin{cases} (0,0,...,0,m) & if\ k = p^2, p \in \mathbb{N} \\ (0,0,....0,0) & otherwise \end{cases}$$

Let $\mathbf{0}$ denote the element $(0,0,....0,0) \in \mathbb{R}^m$ and (ρ_n) be the sequence defined as $\rho_n = n,\ n \in \mathbb{N}$. Then for any $\varepsilon > 0$ and $\varphi \in (0,1]$,

$$\{k \in \mathbb{N} : \mathscr{A}_{||s_k - \mathbf{0}||_{\mathscr{G}}}(\varphi) \geq \varepsilon\} \subseteq \{1,4,9,..\}$$

and eventually $\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||s_k - \mathbf{0}||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}) = 0$. In other words, $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} \mathbf{0}$ in $\mathbb{R}^m$. But it is clear from the definition of s_k that (s_k) is not gradually convergent to $\mathbf{0}$.

Theorem 2.4. *Let (s_k) be any sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$ such that $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ in X. Then s is uniquely determined.*

Proof. If possible suppose $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ and $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s'$ for some $s \neq s'$ in X. Then we have, for any $\varepsilon > 0$ and $\varphi \in (0,1]$, $\delta_\rho(B_1(\varphi,\varepsilon)) = \delta_\rho(B_2(\varphi,\varepsilon)) = 1$ where $B_1(\varphi,\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) < \varepsilon\}$ and $B_2(\varphi,\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s'||_{\mathscr{G}}}(\varphi) < \varepsilon\}$. Choose $m \in B_1(\varphi,\varepsilon) \cap B_2(\varphi,\varepsilon)$, then $\mathscr{A}_{||s_m - s||_{\mathscr{G}}}(\varphi) < \varepsilon$ and $\mathscr{A}_{||s_m - s'||_{\mathscr{G}}}(\varphi) < \varepsilon$. Hence $\mathscr{A}_{||s - s'||_{\mathscr{G}}}(\varphi) \leq \mathscr{A}_{||s_m - s||_{\mathscr{G}}}(\varphi) + \mathscr{A}_{||s_m - s'||_{\mathscr{G}}}(\varphi) < \varepsilon + \varepsilon = 2\varepsilon$. Since ε is arbitrary, so $\mathscr{A}_{||s - s'||_{\mathscr{G}}}(\varphi) = \mathscr{A}_{\bar{0}}(\varphi)$ and we must have $s = s'$. □

Theorem 2.5. *Let (s_k) and (t_k) be two sequences in the GNLS $(X, ||\cdot||_{\mathscr{G}})$ such that $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ and $t_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} t$. Then*
(i) $s_k + t_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s + t$ and (ii) $\lambda s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} \lambda s$, for $\lambda \in \mathbb{R}$.

Proof. (i) Suppose $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ and $t_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} t$. Then for given $\varepsilon > 0$, $\delta_\rho(C_1) = \delta_\rho(C_2) = 0$ where $C_1 = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \frac{\varepsilon}{2}\}$ and $C_2 = \{k \in \mathbb{N} : \mathscr{A}_{||y_k - y||_{\mathscr{G}}}(\varphi) \geq \frac{\varepsilon}{2}\}$. Now as the inclusion $(\mathbb{N} \setminus C_1) \cap (\mathbb{N} \setminus C_2) \subseteq \{k \in \mathbb{N} : \mathscr{A}_{||s_k + t_k - s - t||_{\mathscr{G}}}(\varphi) < \varepsilon\}$ holds, so we must have

$$\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||s_k + t_k - s - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}) \leq \delta_\rho(C_1 \cup C_2) = 0;$$

and consequently, $s_k + t_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s + t$.

(ii) If $\lambda = 0$, then there is nothing to prove. So let us assume $\lambda \neq 0$. Then since $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$, we have for given $\varepsilon > 0$, $\delta_\rho(C_1) = 0$ where $C_1 = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \frac{\varepsilon}{|\lambda|}\}$. Now since $\mathscr{A}_{||\lambda s_k - \lambda s||_{\mathscr{G}}}(\varphi) = |\lambda| \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi)$ holds for any $\lambda \in \mathbb{R}$, we must have $C_2 \subseteq C_1$ where $C_2 = \{k \in \mathbb{N} : \mathscr{A}_{||\lambda s_k - \lambda s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$, which as a consequence implies $\delta_\rho(C_2) = 0$. This completes the proof. □

Theorem 2.6. *Let (s_k) be any sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. If every subsequence of (s_k) is gradual ρ-statistical convergent to s, then (s_k) is also gradual ρ-statistical convergent to s.*

Proof. If possible suppose (s_k) is not gradual ρ-statistical convergent to s. Then there exists some $\varepsilon > 0$ and $\varphi \in (0, 1]$ such that $\delta_\rho(C) \neq 0$, where $C = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$. Since ρ-density of a finite set is zero, so C must be an infinite set. Let $C = \{k_1 < k_2 < ... < k_j < ...\}$. Now define a sequence (t_j) as $t_j = s_{k_j}$ for $j \in \mathbb{N}$. Then (t_j) is a subsequence of (s_k) which is not gradual ρ-statistical convergent to s, a contradiction. □

Remark 2.7. *The converse of the above theorem is not true. One can easily verify this from Example 2.3.*

Theorem 2.8. *Let $(X, ||\cdot||_{\mathscr{G}})$ be a GNLS. A subsequence (s_{k_j}) of a sequence $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ converges to s if and only if $\delta_\rho(K) = 1$, where $K = \{k_j : j \in \mathbb{N}\}$.*

Proof. For any $\varphi \in (0, 1]$ and $\varepsilon > 0$, define the sets:

$$M_\varepsilon = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\} \quad \text{and} \quad M'_\varepsilon = \{k_j \in K : \mathscr{A}_{||s_{k_j} - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$$

Note that $M'_\varepsilon = K \cap M_\varepsilon$.

Necessity: Let $(s_{k_j}) \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$, which implies $\delta_\rho(M'_\varepsilon) = 0$. Since this convergence property must hold and then $M_\varepsilon \subseteq K^c$, which implies, $M_\varepsilon \cap K^c = M_\varepsilon$ Taking the ρ-statistical density on both sides, $\delta_\rho(M_\varepsilon \cap K^c) = \delta_\rho(M_\varepsilon)$ By the hypothesis that the original sequence converges, we have $\delta_\rho(M_\varepsilon) = 0$. Since M_ε is arbitrary then we have $\delta_\rho(M_\varepsilon \cap K^c) = 0$ implies $\delta_\rho(K^c) = 0$. Therefore, by the properties of complementary sets, we conclude

$$\delta_\rho(K) = 1 - \delta_\rho(K^c) = 1 - 0 = 1.$$

Sufficiency: Assume $\delta_\rho(K) = 1$. Since $M'_\varepsilon = K \cap M_\varepsilon \subseteq M_\varepsilon$, it follows that for every $n \in \mathbb{N}$, the counting functions satisfy $|M'_\varepsilon(n)| \leq |M_\varepsilon(n)|$. Multiplying by $\frac{1}{\rho_n}$ and taking the limit then we have,

$$0 \leq \lim_{n\to\infty} \frac{1}{\rho_n} |M'_\varepsilon(n)| \leq \lim_{n\to\infty} \frac{1}{\rho_n} |M_\varepsilon(n)| = \delta_\rho(M_\varepsilon) = 0.$$

Therefore, $\lim\limits_{n\to\infty} \frac{1}{\rho_n} |M'_\varepsilon(n)| = 0$, that implies $\delta_\rho(M'_\varepsilon) = 0$. Hence, $(s_{k_j}) \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$. □

Theorem 2.9. *Let (s_k) and (t_k) be two sequences in the GNLS $(X, ||\cdot||_{\mathscr{G}})$ such that (t_k) is gradually convergent and $\delta_\rho(\{k \in \mathbb{N} : s_k \neq t_k\}) = 0$. Then (s_k) is gradual ρ-statistical convergent.*

Proof. Suppose $\delta_\rho(\{k \in \mathbb{N} : s_k \neq t_k\}) = 0$ holds and $t_k \xrightarrow{||\cdot||_{\mathscr{G}}} t$. Then for every $\varepsilon > 0$ and $\varphi \in (0, 1]$, $\{k \in \mathbb{N} : \mathscr{A}_{||t_k - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$ is a finite set and therefore

$$\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||t_k - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}) = 0. \tag{2.1}$$

Now since the inclusion

$$\{k \in \mathbb{N} : \mathscr{A}_{||s_k - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : \mathscr{A}_{||t_k - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\} \cap \{k \in \mathbb{N} : s_k \neq t_k\}$$

holds, so using Equation (2.1) and the hypothesis we get

$$\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||s_k - t||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}) = 0.$$

Hence $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} t$ and the proof is complete. □

In ([15], Theorem 3.5), Ettefagh et al. proved that in a GNLS every gradual convergent sequence is gradually bounded. But this result is not true in the case of gradual ρ-statistical convergence. Put $\rho_n = n, n \in \mathbb{N}$ and consider the gradual normed space $(\mathbb{R}^2, ||\cdot||_{\mathscr{G}})$ with the gradual norm defined in Example 1.4. Consider the sequence (s_k) in $\mathbb{R}^2$ defined as

$$s_k = \begin{cases} (0,k), & if\ k = p^2, p \in \mathbb{N}; \\ (0,0), & otherwise. \end{cases}$$

Then it is clear that (s_k) is not gradually bounded but gradually ρ-statistical convergent to $(0,0) \in \mathbb{R}^2$.

Definition 2.10. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be gradual ρ-statistical bounded if for every $\varphi \in (0,1]$, there exists $M(= M(\varphi)) > 0$ such that $\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) > M\}) = 0$.*

Theorem 2.11. *Let $(X, ||\cdot||_{\mathscr{G}})$ be a GNLS and suppose (s_k) is a gradual ρ-statistical convergent sequence in X. Then (s_k) is gradual ρ-statistical bounded.*

Proof. Suppose that (s_k) is gradual ρ-statistical convergent to $s \in X$. By Definition 2.1, for any given $\varphi \in (0,1]$ and any $\varepsilon > 0$, the set

$$K_\varepsilon = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}$$

has ρ-density zero. That is, for $K_\varepsilon(n) = \{k \leq n : k \in K_\varepsilon\}$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n}|K_\varepsilon(n)| = 0. \tag{2.2}$$

Then, $\mathscr{A}_{||\cdot||_{\mathscr{G}}}(\varphi)$, we have the following bound for all $k \in \mathbb{N}$,

$$\mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) \leq \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) + \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi).$$

Now, fix $\varepsilon = 1$ and choose a bounding constant M such that $M > 1 + \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi)$. Let us define the set,

$$B_M = \{k \in \mathbb{N} : \mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) \geq M\}.$$

For any $k \in B_M$,

$$M \leq \mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) \leq \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) + \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi).$$

Then we find,

$$\mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq M - \mathscr{A}_{||s||_{\mathscr{G}}}(\varphi) > 1 = \varepsilon.$$

This implies that if $k \in B_M$, then k must also belong to $K_1 = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq 1\}$. Thus, we have

$$\begin{aligned} & B_M \subseteq K_1 \\ \implies & |B_M(n)| \leq |K_1(n)| \\ \implies & 0 \leq \frac{1}{\rho_n}|B_M(n)| \leq \frac{1}{\rho_n}|K_1(n)| \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides and substituting the hypothesis limit from (2.2) for $\varepsilon = 1$, we get:

$$0 \le \lim_{n\to\infty} \frac{1}{\rho_n}|B_M(n)| \le \lim_{n\to\infty} \frac{1}{\rho_n}|K_1(n)| = 0.$$

It follows that,

$$\lim_{n\to\infty} \frac{1}{\rho_n}|B_M(n)| = 0.$$

Therefore, the ρ-density of the set $\{k \in \mathbb{N} : \mathscr{A}_{||s_k||_{\mathscr{G}}}(\varphi) \ge M\}$ is zero. By Definition 1.6, the sequence (s_k) is gradual ρ-statistical bounded. □

Theorem 2.12. *Let (s_k) be a sequence in the generalized normed linear space (GNLS) $(X, ||\cdot||_{\mathscr{G}})$ and let $\rho = (\rho_n)$ be a sequence of positive real numbers satisfying $H = \sup_n \frac{\rho_n}{n} < \infty$. If $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$, then $s_k \xrightarrow{st - ||\cdot||_{\mathscr{G}}} s$.*

Proof. Let $\varepsilon > 0$ be arbitrary,

$$K_\varepsilon = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \ge \varepsilon\}$$

For each $n \in \mathbb{N}$, let $K_\varepsilon(n) = \{k \le n : k \in K_\varepsilon\}$. The cardinality of this truncated set is denoted by $|K_\varepsilon(n)|$. By the hypothesis that $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$, the ρ-statistical density of K_ε is zero. Therefore,

$$\lim_{n\to\infty} \frac{1}{\rho_n}|K_\varepsilon(n)| = 0. \tag{2.3}$$

Given that $H = \sup_n \frac{\rho_n}{n}$, it holds by definition that $\frac{\rho_n}{n} \le H$ for every $n \in \mathbb{N}$. Taking the reciprocal the inequality:

$$\frac{1}{n} \le \frac{H}{\rho_n} \tag{2.4}$$

Since $|K_\varepsilon(n)| \ge 0$, multiplying both sides of (2.4) by the cardinality of the subset preserves the inequality, establishing that:

$$0 \le \frac{1}{n}|K_\varepsilon(n)| \le \frac{H}{\rho_n}|K_\varepsilon(n)| \tag{2.5}$$

Taking the limit as $n \to \infty$ across the inequality in (2.5), we obtain:

$$0 \le \lim_{n\to\infty} \frac{1}{n}|K_\varepsilon(n)| \le H \cdot \lim_{n\to\infty} \frac{1}{\rho_n}|K_\varepsilon(n)|$$

Applying our hypothesis limit from (2.3) to the right-hand bound gives:

$$0 \le \lim_{n\to\infty} \frac{1}{n}|K_\varepsilon(n)| \le H \cdot 0 = 0$$

It follows that:

$$\lim_{n\to\infty} \frac{1}{n}|K_\varepsilon(n)| = 0$$

This implies that the natural statistical density of K_ε is zero (i.e., $\delta(K_\varepsilon) = 0$). Hence, $s_k \xrightarrow{st - ||\cdot||_{\mathscr{G}}} s$. □

The converse of the above theorem is not necessarily true. The following example clarify this fact.

Example 2.13. *Let $X = \mathbb{R}^m$ and let $\|\cdot\|_{\mathscr{G}}$ be the generalized norm defined in Example 1.4. Let (ρ_n) be a sequence of positive real numbers satisfying $\lim_{n\to\infty} \rho_n \to \infty$ and $\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\rho_n} \to \infty$. Consider the sequence (s_k) in $\mathbb{R}^m$ defined by*

$$s_k = \begin{cases} (0,0,\ldots,0,\rho_n), & \text{if } k = n^3 \text{ for some } n \in \mathbb{N}, \\ (0,0,\ldots,0,0), & \text{otherwise.} \end{cases}$$

We shall show that $s_k \xrightarrow{st-\|\cdot\|_{\mathscr{G}}} \mathbf{0}$ but (s_k) is not gradual ρ-statistical convergent to $\mathbf{0}$.
***Justification**: First, let $\varepsilon > 0$ be arbitrary. We examine the standard natural statistical density of the set of indices where the generalized norm of s_k deviates from $\mathbf{0}$. We have*

$$K_\varepsilon(n) = \{k \le n : \mathscr{A}_{\|s_k - \mathbf{0}\|_{\mathscr{G}}}(\varphi) \ge \varepsilon\}$$

Since $s_k \neq \mathbf{0}$ only when k is a perfect cube, at most the number of perfect cubes less than or equal to n. Thus, for any $\varepsilon > 0$, we have

$$|K_\varepsilon(n)| \le |\{k \le n : k = m^3 \text{ for some } m \in \mathbb{N}\}| = \lfloor \sqrt[3]{n} \rfloor \le \sqrt[3]{n}.$$

Therefore,

$$0 \le \lim_{n\to\infty} \frac{1}{n}|K_\varepsilon(n)| \le \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n} = \lim_{n\to\infty} \frac{1}{n^{2/3}} = 0.$$

That implies, $\delta(K_\varepsilon) = 0$, which proves that $s_k \xrightarrow{st-\|\cdot\|_{\mathscr{G}}} \mathbf{0}$. Next, we demonstrate that (s_k) fails to converge gradual ρ-statistically to $\mathbf{0}$. Choose a specific threshold $\varepsilon_0 = e^{\varphi}$ for $\varphi \in (0,1]$ such that the membership function $\mathscr{A}_{\|s_k - \mathbf{0}\|_{\mathscr{G}}}(\varphi) \ge \varepsilon_0$ is satisfied for all non-zero terms s_k where $k = n^3$. For a given $n \in \mathbb{N}$, and the number of perfect cubes less than or equal to n, which is given by $\lfloor \sqrt[3]{n} \rfloor$. We can express this value as $\lfloor \sqrt[3]{n} \rfloor = \sqrt[3]{n} - t_n$, where $0 \le t_n < 1$. Evaluating the ρ-statistical density weight for this set, we have

$$\frac{1}{\rho_n}|\{k \le n : \mathscr{A}_{\|s_k - \mathbf{0}\|_{\mathscr{G}}}(\varphi) \ge \varepsilon_0\}| = \frac{1}{\rho_n}\lfloor \sqrt[3]{n} \rfloor = \frac{\sqrt[3]{n}}{\rho_n} - \frac{t_n}{\rho_n}.$$

Taking the limit as $n \to \infty$ on both sides, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n}|\{k \le n : \mathscr{A}_{\|s_k - \mathbf{0}\|_{\mathscr{G}}}(\varphi) \ge \varepsilon_0\}| = \infty - 0 = \infty \neq 0.$$

Since the limit is non-zero, the ρ-statistical density $\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{\|s_k - \mathbf{0}\|_{\mathscr{G}}}(\varphi) \ge \varepsilon_0\}) \neq 0$. Consequently, (s_k) is not gradual ρ-statistical convergent to $\mathbf{0}$.

Theorem 2.14. *Let (s_k) be a sequence in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$ such that $s_k \xrightarrow{st-\|\cdot\|_{\mathscr{G}}} s$. Then, $s_k \xrightarrow{st_\rho-\|\cdot\|_{\mathscr{G}}} s$ holds if $\inf_n \frac{\rho_n}{n} > 0$.*

Proof. The proof easily follows from the following inequation

$$\frac{1}{n} \mid \{k \in \mathbb{N} : \mathscr{A}_{\|s_k - s\|_{\mathscr{G}}}(\varphi) \ge e^{\varphi}\} \mid \ge (\inf_n \frac{\rho_n}{n}) \cdot \frac{1}{\rho_n} \mid \{k \in \mathbb{N} : \mathscr{A}_{\|s_k - s\|_{\mathscr{G}}}(\varphi) \ge e^{\varphi}\} \mid .$$

□

Corollary 2.15. *A necessary and sufficient condition for a gradual statistical convergent sequence (s_k) in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$ to be gradual ρ-statistical convergent is that $\inf_n \frac{\rho_n}{n} > 0$.*

Definition 2.16. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be strongly gradual ρ-summable to s in X if*

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) = 0.$$

Theorem 2.17. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. If (s_k) is strongly gradual ρ-summable to s then $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$.*

Proof. Let (s_k) be strongly gradual ρ-summable to s in X. Then, by definition

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) = 0 \tag{2.6}$$

holds and eventually, we have

$$\begin{aligned} \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) &= \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)\geq\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) + \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)<\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \\ &\geq \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)\geq\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \\ &\geq \frac{\varepsilon}{\rho_n} \left| \left\{ k \leq n : \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \geq \varepsilon \right\} \right| . \end{aligned}$$

Letting $n \to \infty$ and using Equation (2.6), we obtain

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left| \left\{ k \leq n : \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \geq \varepsilon \right\} \right| = 0.$$

Hence, $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$. □

Theorem 2.18. *Let (s_k) be a gradually bounded sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. If $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$ then (s_k) is strongly gradual ρ-summable to s.*

Proof. Since $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$, so for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left| \left\{ k \leq n : \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \geq \varepsilon \right\} \right| = 0. \tag{2.7}$$

Now since (s_k) is gradually bounded, such that $\forall k \in \mathbb{N}$ and for any $\varphi \in (0, 1]$ and for any $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) &= \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)\geq\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) + \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)<\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \\ &\geq \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)\geq\varepsilon}}^{n} 1 + \frac{1}{\rho_n} \sum_{\substack{k=1 \\ \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi)<\varepsilon}}^{n} \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \\ &\geq \frac{1}{\rho_n} \left| \left\{ k \leq n : \mathscr{A}_{||s_k-s||_{\mathscr{G}}}(\varphi) \geq \varepsilon \right\} \right| + \varepsilon \cdot \frac{n}{\rho_n}. \end{aligned}$$

From Equation (2.7) and the above inequation we have

$$\lim_{n\to\infty} \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) = 0.$$

Hence, (s_k) is strongly gradual ρ-summable to s. This completes the proof. □

Definition 2.19. *Let (s_k) be a sequence in the GNLS $(X, ||\cdot||_{\mathscr{G}})$. Then (s_k) is said to be gradual ρ-statistical Cauchy if for every $\varepsilon > 0$ and $\varphi \in (0,1]$, there exists a natural number $N(= N(\varphi,\varepsilon))$ such that $\delta_\rho(\{k \in \mathbb{N} : \mathscr{A}_{||s_k - s_N||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}) = 0$.*

Theorem 2.20. *Let $(X, ||\cdot||_{\mathscr{G}})$ be a GNLS. Then every gradual ρ-statistical convergent sequence in X is a gradual ρ-statistical Cauchy sequence.*

Proof. Let (s_k) be a sequence in X such that $s_k \xrightarrow{st_\rho - ||\cdot||_{\mathscr{G}}} s$. Then for every $\varepsilon > 0$ and $\varphi \in (0,1]$,

$$\delta_\rho(B_1(\varphi,\varepsilon)) = 0 \text{ where } B_1(\varphi,\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s||_{\mathscr{G}}}(\varphi) \geq \varepsilon\}. \tag{2.8}$$

Choose $N \in \mathbb{N} \setminus B_1(\varphi,\varepsilon)$. Then we have $\mathscr{A}_{||s_N - s||_{\mathscr{G}}}(\varphi) < \varepsilon$.

Let $B_2(\varphi,\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}_{||s_k - s_N||_{\mathscr{G}}}(\varphi) \geq 2\varepsilon\}$. Now we prove that the following inclusion is true

$$B_2(\varphi,\varepsilon) \subseteq B_1(\varphi,\varepsilon). \tag{2.9}$$

For if $p \in B_2(\varphi,\varepsilon)$ we have

$$2\varepsilon \leq \mathscr{A}_{||s_p - s_N||_{\mathscr{G}}}(\varphi) \leq \mathscr{A}_{||s_p - s||_{\mathscr{G}}}(\varphi) + \mathscr{A}_{||s - s_N||_{\mathscr{G}}}(\varphi) < \mathscr{A}_{||s_p - s||_{\mathscr{G}}}(\varphi) + \varepsilon,$$

which implies $p \in B_1(\varphi,\varepsilon)$ and so Equation (2.9) is true. From Equation (2.8) and Equation (2.9) we conclude that $\delta_\rho(B_2(\varphi,\varepsilon)) = 0$ which means that (s_k) is a gradual ρ-statistical Cauchy sequence. □

3 Conclusions

In this study, we have investigated several core characteristics of ρ-statistical convergence in gradual normed linear spaces. Additionally, we have introduced the notion of strong ρ-statistical summability in these spaces and demonstrated that every strongly gradual ρ-summable sequence exhibits gradual ρ-statistical convergence. Lastly, we have defined the concept of ρ-statistical Cauchy sequences in gradual normed spaces and elucidated the connection between gradual ρ-statistical convergence and gradual ρ-statistical Cauchy sequences.

Summability theory and the convergence of sequences play significant roles across different branches of mathematics, notably in mathematical analysis. However, research focusing on gradual normed linear spaces in this context is relatively nascent and hasn't garnered extensive attention yet. The findings presented here could serve as a starting point for future investigations, allowing researchers to delve deeper into the diverse concepts of convergence in gradual normed linear spaces.

Data Availability Statement

This article has no associated data.

Conflict of interest

This article has no conflict of interest.

References

[1] C. Belen and S. A. Mohiuddine, Generalized weighted statistical convergence and application, *Appl. Math. Comput.* **219** (2013), no. 18, 9588–9594. DOI: https://doi.org/10.1016/j.amc.2013.03.115

[2] V. K. Bhardwaj, S. Dhawan and S. Gupta, Density by moduli and statistical boundedness, *Abstr. Appl. Anal.* **2016** (2016), Art. ID 2143018, 8 pp. DOI: https://doi.org/10.1155/2016/2143018

[3] V. K. Bhardwaj and S. Gupta, On some generalizations of statistical boundedness, *J. Inequal. Appl.* **2014** (2014), no. 12, 11 pp. DOI: https://doi.org/10.1186/1029-242X-2014-12

[4] H. Cakalli, Lacunary statistical convergence in topological groups, *Indian J. Pure Appl. Math.* **26** (1995), no. 2, 113–119.

[5] H. Cakalli, A study on statistical convergence, *Funct. Anal. Approx. Comput.* **1** (2009), no. 2, 19–24.

[6] H. Cakalli, A variation on statistical ward continuity, *Bull. Malays. Math. Sci. Soc.* **40** (2017), no. 4, 1701–1710. DOI: https://doi.org/10.1007/s40840-015-0195-0

[7] C. Choudhury and S. Debnath, On $\mathcal{I}$-convergence of sequences in gradual normed linear spaces, *Facta Univ. Ser. Math. Inform.* **36** (2021), no. 3, 595–604. DOI: https://doi.org/10.22190/FUMI210108044C

[8] C. Choudhury and S. Debnath, On quasi statistical convergence in gradual normed linear spaces, *TWMS J. App. Eng. Math.* **13** (2023), no. 4, 1327–1336.

[9] C. Choudhury and S. Debnath, On λ-statistical convergence of sequences in gradual normed spaces, *Appl. Math. E-Notes* **23** (2023), 209–219.

[10] D. Dubois and H. Prade, Gradual elements in a fuzzy set, *Soft Comput.* **12** (2008), no. 2, 165–175. DOI: https://doi.org/10.1007/s00500-007-0187-6

[11] A. Esi, S. Araci and M. Acikgoz, Statistical convergence of Bernstein operators, *Appl. Math. Inf. Sci.* **10** (2016), no. 6, 2083–2086. DOI: https://doi.org/10.18576/amis/100610

[12] M. Et, V. K. Bhardwaj and S. Gupta, On deferred statistical boundedness of order α, *Comm. Statist. Theory Methods* **51** (2022), no. 24, 8786–8798. DOI: https://doi.org/10.1080/03610926.2021.1906434

[13] M. Et, S. A. Mohiuddine and H. Şengül, On lacunary statistical boundedness of order α, *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 3, 707–716.

[14] M. Ettefagh, F. Y. Azari and S. Etemad, On some topological properties in gradual normed spaces, *Facta Univ. Ser. Math. Inform.* **35** (2020), no. 3, 549–559.

[15] M. Ettefagh, S. Etemad and F. Y. Azari, Some properties of sequences in gradual normed spaces, *Asian-Eur. J. Math.* **13** (2020), no. 4, Art. ID 2050085, 12 pp. DOI: https://doi.org/10.1142/S1793557120500850

[16] H. Fast, Sur la convergence statistique, *Colloq. Math.* **2** (1951), no. 3-4, 241–244.

[17] J. Fortin, D. Dubois and H. Fargier, Gradual numbers and their application to fuzzy interval analysis, *IEEE Trans. Fuzzy Syst.* **16** (2008), no. 2, 388–402. DOI: https://doi.org/10.1109/TFUZZ.2006.890680

[18] J. A. Fridy, On statistical convergence, *Analysis* **5** (1985), no. 4, 301–314. DOI: https://doi.org/10.1524/anly.1985.5.4.301

[19] J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.* **125** (1997), no. 12, 3625–3631. DOI: https://doi.org/10.1090/S0002-9939-97-04000-8

[20] M. Kucukaslan and M. Yilmazturk, On deferred statistical convergence of sequences, *Kyungpook Math. J.* **56** (2016), no. 2, 357–366.

[21] L. Lietard and D. Rocacher, Conditions with aggregates evaluated using gradual numbers, *Control Cybernet.* **38** (2009), no. 2, 395–417.

[22] M. Mursaleen, λ-statistical convergence, *Math. Slovaca* **50** (2000), no. 1, 111–115.

[23] M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.* **288** (2003), no. 1, 223–231. DOI: https://doi.org/10.1016/j.jmaa.2003.08.004

[24] I. Sadeqi and F. Y. Azari, Gradual normed linear space, *Iran. J. Fuzzy Syst.* **8** (2011), no. 5, 131–139. DOI: https://doi.org/10.22111/ijfs.2011.302

[25] E. Savas, On statistically convergent double sequences of fuzzy numbers, *Inf. Sci.* **162** (2004), no. 3-4, 183–192. DOI: https://doi.org/10.1016/j.ins.2003.09.005

[26] E. Savas and P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.* **24** (2011), no. 6, 826–830. DOI: https://doi.org/10.1016/j.aml.2010.12.022

[27] B. C. Tripathy, Statistically convergent double sequences, *Tamkang J. Math.* **34** (2003), no. 3, 231–237. DOI: https://doi.org/10.5556/j.tkjm.34.2003.314

[28] L. A. Zadeh, Fuzzy sets, *Inf. Control* **8** (1965), no. 3, 338–353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

[29] B. Baxhaku, P. Agrawal and R. Shukla, Some fuzzy Korovkin type approximation theorems via power series summability method, *Soft Comput.* **26** (2022), no. 21, 11501–11507. DOI: https://doi.org/10.1007/s00500-022-07429-6